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Drag on a Body in Nearly-Free Molecular Flow

MARIAN H. ROSE

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Abstract

A linearized Krook equation with the addition of a point source term is used to find the steady flow field in the vicinity of an object in nearly-free molecular flight through a non-ionized gas. The point source is related to the shape of the object and to the boundary conditions both at its surface and at infinity.

A knowledge of the flow field and thence of the distribution function near the body permits the drag to be determined as a first order perturbation on the free-molecular value.

The cases of a plate and of a sphere undergoing specular and diffuse reflection, respectively, are worked out in detail. For specular reflection, the drag on a plate is evaluated for all Mach numbers; for diffuse, due to the length of the calculation, only the case of a sphere at high Mach numbers is considered. The results, in both cases, are valid to first order in the Knudsen number.

Acknowledgments

The author is most grateful to Professor Harold Grad for having suggested this problem and for his continued interest throughout the work. She also wishes to thank Professor Harold Weitzner for some helpful discussions.

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Drag on a Body in Nearly-Free Molecular Flow

I. Introduction

The region of nearly-free molecular gas flow, i.e., the transition region from free flow to the start of intermolecular collisions, is analyzed in this report with a view to computing the drag on an object traversing a rarefied atmosphere at a constant speed. The method to be presented departs from those previously used for this problem, namely, the "first collision" and the "integral equation" methods which we shall now describe briefly.

In the "first collision" method [1-7] it is assumed that the most important correction to the free molecular regime arises from collisions between free stream particles that have been reflected from the object and those that have not. Other types of collisions are disregarded as being of second order. Baker and Charwat [5] have analyzed the drag on a sphere using this and additional assumptions, which are:

1) Billiard-ball type molecules.

2) The molecules reflected from the surface are assigned an average speed $\overline{V}_e$ and their velocity distribution is disregarded.

3) The velocity of flight V_o is sufficiently large for the velocity distribution of the incoming molecules to be disregarded.

4) $\overline{V}_e/V_o << 1$.

5) The mean free path of the emitted beam relative to the

incoming is constant.

6) The energy accommodation coefficient is unity and the body is cold.

The drag coefficient for a sphere is

$$c_d = 2[1 + 0.44 \frac{\overline{V}_e}{V_o} - (0.46 \frac{V_o}{\overline{\overline{V}}_e} + 1.3)a\rho_o S_{eo}] \tag{I.1}$$

where $\underline{a}$ is the radius of the sphere, ρ_o the number density of the oncoming stream, and S_{eo} the collision cross-section of the emitted stream with respect to the oncoming one.

For a flat plate oriented perpendicularly to the flow, Baker and Charwat obtain

$$c_D = 2[1 + \frac{2}{3} \frac{\overline{V}_e}{V_o} - 0.32\ a\rho_o \frac{V_o}{\overline{\overline{V}}_e} S_{eo} - 1.36\ \rho_o a\ S_{eo}] \ . \tag{I.2}$$

Using similar assumptions but slightly different computational methods, Hammerling and Kivel [6] arrive at the following expression for the energy imparted to a sphere in unit time:

$$E = \pi a^2 \frac{1}{2} \rho_o m_o V_o^2 \left[1 - \left(\frac{\overline{V}_e}{V_o} \right)^2 \right] (1 - 0.2 \frac{V_o}{\overline{\overline{V}}_e} a\rho_o S_{eo}) \ . \tag{I.3}$$

Comparison shows that $0.44\ \overline{V}_e/V_o$ in Baker and Charwat's c_d is replaced by $(1 - \overline{V}_e/V_o)^2$ in Hammerling and Kivel's, and that $0.46\ V_o/\overline{\overline{V}}_e\ a\rho_o S_{eo}$ is replaced by $0.2\ V_o/\overline{\overline{V}}_e\ a\rho_o S_{eo}$. These large differences demonstrate the sensitivity of the results to the approximations used in the numerical work. For example,

the distribution of emitted molecules is approximated by slightly different functions in each case; the resulting differences in the coefficients is due to the fact that they represent the small net between the gain and loss of molecules hitting the sphere due to collisions. These results, although semi-quantitative, predict trends which agree with experiments [7,8].

In order to analyze the range of validity of some of the assumptions made by Baker and Charwat, Willis [9] has integrated the Boltzmann equation formally by writing it as a pure integral equation and then solving it by successive approximations. The series thus obtained has been proved convergent for the special case of linearized Couette flow [10]; furthermore, it has always led to physically reasonable results. By approximating the Boltzmann collision term by a modified Kröok term Willis has shown that:

1) The collisions between streams of reflected particles have only a small effect on the drag compared with the first collisions. However, this effect increases as $\overline{V}_e/V_o$ decreases.

2) Extrapolation of the first collision results to higher values of $\overline{V}_e/V_o$ is reasonable for pseudo-hard sphere molecules, but is not justifiable for pseudo-Maxwellian molecules. Thus, the choice of a molecular model is of great importance if the energy accommodation is small and, therefore, the reflected molecules are no longer cold.

3) The qualitative agreement between the two methods improves as $\overline{V}_e/V_o \rightarrow 0$.

A third method, as yet not fully analyzed, has been advanced by Enoch [11]. It is applicable to high speed flows for which the distribution functions, which are highly peaked about some average velocity, may be approximated by δ-functions. The full collision integral is then replaced by a simpler model which, however, retains the three conservation laws. The foregoing analysis has been successfully applied to high speed plane Couette flow, the distribution function and its first three moments having been obtained analytically.

The method described in this report is likewise based on replacing the Boltzmann equation by a simpler model equation which yet retains its essential features. The equation chosen is the "single-relaxation time" Boltzmann equation or "Krook equation" developed by Bhatnager, Gross and Krook [12] and which preserves the three conservation laws. The same equation was used by Grad [13] in investigating the asymptotic behavior of the flow far from the source. In the asymptotic case, however, the drag could not be found since the source function was unknown. A knowledge of this function can be obtained only by determining the flow in the vicinity of the object (less than one mean free path away) and yet at a sufficient distance for the object to be replaceable by a point source. This is the region to be studied in this report.

In the limit of a point source and granted the validity of the Krook equation we obtain, in this report, the first order perturbation on the free flow. This result is rigorous, i.e., it

encompasses all collisions between particles and not merely "first collisions."

Our results are obtained by following the method used by Grad for the asymptotic flow. Briefly, we linearize the Krook equation about the Maxwellian distribution at infinity and then find the Fourier transform of the perturbed distribution function. This yields the exact flow field everywhere in terms of the Fourier transforms. The inversion is different, however, in the two cases; in the asymptotic case the mean free path is assumed much smaller than the observer distance, whereas in ours, it is much larger.

These methods are applied to finding the drag on a sphere (and concurrently on a plate) travelling at a steady speed through a gas at rest. Calculations for two different boundary conditions are made, namely, specular reflection and diffuse reflection. In both cases it is possible in principle to find the drag for any Mach number. However, due to the length of the computations, only the following cases are carried through:

a) the drag on a plate moving in a direction perpendicular to its plane for all Mach numbers with specular reflection at the boundary,

b) the drag on a sphere at high Mach numbers only with diffuse reflection at the boundary.

We obtain the following corrections to the drag:

$$D^{(1)} = -\frac{4A\tilde{u}_o}{(2\pi)^2(2\pi)^{3/2}}\frac{a}{L}\left\{\tilde{u}_o(k_1 + 12k_2)e^{-\frac{1}{2}\tilde{u}_o^2} + [k_1(1+\tilde{u}_o^2)\right.$$

$$\left. + 2k_2]\int_0^{\tilde{u}_o} e^{-\frac{1}{2}x^2}dx - 2\tilde{u}_o(k_1 + 8k_2)\right\}$$

for the plate, at any Mach number undergoing specular reflection,

$$D^{(1)} = \frac{a}{L}A\frac{\gamma^2\tilde{u}_o^3}{8\sqrt{2}}$$

for the sphere at high Mach number, undergoing diffuse reflection,

where

$\tilde{u}_o$ = Mach number

A = cross-sectional area of the sphere or area of the plate,

a = radius of plate or sphere,

L = mean free path,

$k_1 = -16\sqrt{\pi}$,

$k_2 = \frac{32\pi^2}{3}$,

γ = ratio of temperature of incoming stream to reflected stream.

II. Basic Assumptions

Our aim is to determine the drag on a body travelling at constant speed u_o through a rarefied gas at rest.

There are three lengths of interest in this problem: the mean free path L, the characteristic body dimension a, and the observer distance R from the object. We shall assume that $R/a \gg 1$ and that $L/R \gg 1$ which together imply that $L/a \gg 1$. The first assumption was also made by Grad in investigating the asymptotic steady flow past an object; thus it was possible to replace the body by a point source function, the value of the drag being directly related to this function. However, the source function itself could not be determined because Grad assumed $L/R \ll 1$, i.e., only asymptotic solutions were sought for which the observer was many mean free paths away from the object. However, Grad's results are valid for any value of the Knudsen number L/a, an interesting conclusion. This is not true for our case.

In order to determine the source function and thence the drag, we must know the distribution function f in the neighborhood of the body, i.e., within a mean free path distance ($L/R \gg 1$). However, we do not wish to go too close to the body because this is essentially the region of free-molecular flow where the effect of collisions is, as yet, unimportant. Now, it is precisely the effect of these intra-molecular collisions on the drag that we wish to determine, and so we must assume that $R/a \gg 1$.

This last assumption allows us to use Grad's procedure for linearizing the Krook equation, and we obtain the same moment equations. However, the evaluation of the moments by Fourier transform techniques and the subsequent inversions are radically different in the two theories because of the different ranges of values for L/R. The linearization procedure and the evaluation of the Fourier transforms will be described in detail in the following sections.

In order to treat some specific problems we shall consider the cases of a plate moving in a direction perpendicular to its plane and of a sphere. In the former case we shall assume specular reflection at the surface and in the latter, diffuse. At infinity, the distribution function will be assumed Maxwellian.

III. Basic Equations

We start from the Krook equation with a point source function, namely,

$$\text{(III.1)} \qquad \xi \cdot \frac{\partial f}{\partial x} = \nu(f_o - f) + \sigma(\xi)\delta(x) \ .$$

Here $f(x,\xi)$ is the distribution function, x is the position vector, ξ the velocity vector, ν the collision frequency, assumed constant, and f_o the locally Maxwellian distribution function given by

$$f_o(x,\xi) = \frac{\bar{\rho}}{(2\pi R\bar{T})^{3/2}} \exp[-(\xi - u)^2/2R\bar{T}] \ .$$

$f_o(x,\xi)$ introduces a non-linear term into equation (III.1) because it depends on f through $\bar{\rho}$, $\bar{T}$, $\bar{u}$:

$$\bar{\rho}(x) = \int f \, d\xi$$

$$\text{(III.2)} \qquad \overline{\rho u} = \int \xi f \, d\xi$$

$$\bar{p} = \bar{\rho}R\bar{T} = \int \frac{1}{3} (\xi - \bar{u})^2 f \, d\xi \ .$$

The source function $\sigma(\xi)$ will be subsequently identified with the net mass flow of the perturbed distribution function through the surface of the body.

Equation (III.1) may be linearized using Grad's method [13] and then Fourier transformed. Although repetitious, this linearization procedure will be re-derived here since it has a

fundamental bearing on our problem.

We shall assume that the distribution function $f^{(o)}$ at infinity is Maxwellian; the corresponding density, velocity, and temperature are denoted by ρ_o, u_o, T_o. Linearizing about these quantities leads to

$$\text{(III.3)} \qquad \begin{cases} f = f^{(o)} + g \\ \bar{\rho} = \rho_o + \rho \\ \bar{u} = u_o + u \\ \bar{T} = T_o + T \, , \end{cases}$$

where

$$f^{(o)} = \frac{\rho_o}{(2\pi RT_o)^{3/2}} \exp[-(\xi - u_o)^2 / 2RT_o] \; .$$

The same linearization applied to f_o yields

$$\text{(III.4)} \qquad f_o = f^{(o)} \left\{ 1 + \frac{\rho}{\rho_o} + \left(\frac{c^2}{2RT_o} - \frac{3}{2}\right) \frac{T}{T_o} + \frac{c \cdot u}{RT_o} \right\}$$

with $c = \xi - u_o$.

Inserting these expressions into (III.1) results in

$$\text{(III.5)} \qquad \xi \cdot \frac{\partial g}{\partial x} + \nu g = \nu f^{(o)} \left\{ \frac{\rho}{\rho_o} + \left(\frac{c^2}{2RT_o} - \frac{3}{2}\right) \frac{T}{T_o} + \frac{c \cdot u}{RT_o} \right\} + \sigma \delta(x)$$

which is linear in g since

$$\text{(III.6)} \qquad \begin{cases} \rho = \int g \, d\xi \\ \rho_o u + \rho u_o = \int \xi \, g \, d\xi \\ \rho_o u = \int c \, g \, d\xi \\ \rho = \int \frac{1}{3} c^2 g \, d\xi \\ \rho_o RT = \int \left(\frac{1}{3} c^2 - RT_o\right) g \, d\xi \; . \end{cases}$$

We now introduce the following dimensionless quantities (except for L and x where $L = 1/\nu$ is the m.f.p.):

$$\tilde{\rho} = \rho/\rho_o \quad , \quad \tilde{T} = T/T_o \quad , \quad \tilde{p} = p/p_o$$

$$\tilde{u} = u/\sqrt{RT_o} \; , \quad \tilde{u}_o = u_o/\sqrt{RT_o}$$

(III.7)
$$\tilde{\xi} = \xi/\sqrt{RT_o} \; , \quad \tilde{c} = c/\sqrt{RT_o} = \tilde{\xi} - \tilde{u}_o$$

$$\tilde{g} = (RT_o)^{3/2} g/\rho_o , \quad \omega = \frac{1}{(2\pi)^{3/2}} \exp(-\tfrac{1}{2}\tilde{c}^2) = (RT_o)^{3/2} f^o/\rho_o$$

$$\frac{1}{L} = \nu/\sqrt{RT_o}$$

Then equation (III.5) becomes

(III.8)
$$\tilde{\xi}\cdot\frac{\partial\tilde{g}}{\partial x} + \frac{1}{L}\tilde{g} = \frac{\omega}{L}\left\{\tilde{\rho} + (\tfrac{1}{2}\tilde{c}^2 - \tfrac{3}{2})\tilde{T} + \tilde{c}\cdot\tilde{u}\right\} + \tilde{\sigma}\delta(x) \; ,$$

and the relations (III.6) take the form

(III.9)
$$\tilde{\rho} = \int \tilde{g}\, d\tilde{\xi} \; , \qquad \tilde{u} = \int \tilde{c}\,\tilde{g}\, d\tilde{\xi}$$
$$\tilde{p} = \int \tfrac{1}{3}\tilde{c}^2\tilde{g}\, d\tilde{\xi} \; , \qquad \tilde{T} = \int (\tfrac{1}{3}\tilde{c}^2 - 1)\tilde{g}\, d\tilde{\xi} \; .$$

For the Fourier transform of (III.8) we have

(III.10)
$$\bar{\tilde{g}}(k,\tilde{\xi}) = \frac{\omega(\tilde{\xi})}{1 + iL\tilde{\xi}\cdot k}\left\{\bar{\tilde{\rho}}(k) + (\tfrac{1}{2}\tilde{c}^2 - \tfrac{3}{2})\bar{\tilde{T}}(k) + \tilde{c}\cdot\bar{\tilde{u}}(k)\right\}$$
$$+ \frac{L\,\tilde{\sigma}(\tilde{\xi})}{1 + iL\tilde{\xi}\cdot k} \; ,$$

where a function $\phi(x)$ and its transform $\bar{\phi}(k)$ are interrelated by

$$\overline{\phi}(k) = \int e^{-ik\cdot x}\, \phi(x)dx$$

and

$$\phi(x) = \frac{1}{(2\pi)^3}\int e^{ik\cdot x}\, \overline{\phi}(k)dk \ .$$

If, now, we multiply (III.11) by 1, $\tilde{c}$, and $(\frac{1}{3}\,\tilde{c}^2 - 1)$ and integrate over velocity space, we obtain a set of five linear, inhomogeneous equations in $\tilde{\overline{\rho}}$, $\tilde{\overline{T}}$, and $\tilde{\overline{u}}$. These are

(III.11)

$$\begin{pmatrix} 1 - \int \Omega\, d\tilde{\xi} & -\int \frac{1}{2}(\tilde{c}^2-3)\Omega d\tilde{\xi} & -\int \tilde{c}_s \Omega d\tilde{\xi} \\ -\int \frac{1}{3}(\tilde{c}^2-3)\Omega d\tilde{\xi} & 1-\int \frac{1}{6}(\tilde{c}^2-3)^2\Omega d\tilde{\xi} & -\int \frac{1}{3}\tilde{c}_s(\tilde{c}^2-3)\Omega d\tilde{\xi} \\ -\int \tilde{c}_r \Omega d\tilde{\xi} & -\int \frac{1}{2}\tilde{c}_r(\tilde{c}^2-3)\Omega d\tilde{\xi} & \delta_{rs} - \int \tilde{c}_r\tilde{c}_s \Omega d\tilde{\xi} \end{pmatrix} \begin{pmatrix} \tilde{\overline{\rho}} \\ \tilde{\overline{T}} \\ \tilde{\overline{u}}_r \end{pmatrix} = L \begin{pmatrix} \tilde{\overline{\sigma}}_1 \\ \tilde{\overline{\sigma}}_2 \\ \tilde{\overline{\sigma}}_3^{(r)} \end{pmatrix}$$

where

$$\text{(III.12)} \quad \begin{cases} \Omega = \dfrac{\omega}{1 + iL\tilde{\xi}\cdot k} \\ \tilde{\sigma}_1(k) = \displaystyle\int \frac{\tilde{\sigma}(\tilde{\xi})d\tilde{\xi}}{1 + iL\tilde{\xi}\cdot k} \\ \tilde{\sigma}_2(k) = \displaystyle\int \frac{((1/3)\tilde{c}^2 - 1)\tilde{\sigma}(\tilde{\xi})d\tilde{\xi}}{1 + iL\tilde{\xi}\cdot k} \\ \tilde{\sigma}_3^{(i)}(k) = \displaystyle\int \frac{\tilde{c}_1\tilde{\sigma}d\tilde{\xi}}{1 + iL\tilde{\xi}\cdot k} \ . \end{cases}$$

The subscripts r and s range from 1 to 3.

The next step is to integrate the various components of (III.11) over $\tilde{\xi}$, solve for $\tilde{\overline{\rho}}$, $\tilde{\overline{T}}$, $\tilde{\overline{u}}_r$, and then perform the

Fourier inversion. If L is small, the denominator of Ω may be expanded in powers of L and the $\tilde{\xi}$-integration and subsequent Fourier inversion of (III.11) presents no special problems. This is the case considered by Grad and subsequently by Sirovich [14]; the observer point is many mean free paths away from the body. In the near region in which we are interested, however, L is large and the $\tilde{\xi}$-integration of the components of (III.11) requires special consideration. Some of these components contain $\sigma(\xi)$ in their integrands and it is, therefore, appropriate to consider the nature of $\sigma(\xi)$. This we shall do in Section V.

IV. Inversion of Equations (III.11)

The 5 x 5 matrix (III.11) is of the form

$$\text{(IV.1)} \qquad \begin{pmatrix} 1-\varepsilon b_{ij} & \varepsilon b_{ij} \\ \varepsilon b_{ij} & 1-\varepsilon b_{ij} \end{pmatrix}_{ij}$$

where

$$i = 1,2,\ldots,5$$

$$j = 1,2,\ldots,5$$

and ε is a small parameter ($\varepsilon \sim 1/L$).

To calculate the determinant of this matrix up to terms in ε^5 would be extremely laborious. Fortunately, due to the form of equations (III.8) it will only be necessary to calculate it to lowest order in ε. For completeness, we shall compute the determinant of (IV.1) to next order in ε. When solving the linear equations (III.11) for $\tilde{\rho}(k)$, $\tilde{T}(k)$, and $\tilde{u}(k)$, we shall encounter 5 x 5 determinants of the form*

$$\begin{vmatrix} x_1 & \varepsilon b_{12} & \varepsilon b_{13} & \varepsilon b_{14} & \varepsilon b_{15} \\ x_2 & 1-\varepsilon b_{22} & \cdots & \cdots & \cdots \\ x_3 & \varepsilon b_{31} & 1-\varepsilon b_{33} & \cdots & \varepsilon b_{35} \\ x_4 & \cdots & \cdots & \cdots & \cdots \\ x_5 & \cdots & \cdots & \cdots & \cdots \end{vmatrix}$$

---

* More precisely, the x_i's may occupy any column but the determinant may be converted to the form shown above by changing rows and columns.

In order to solve this determinant up to $O(\varepsilon)$, it is sufficient to retain the terms in the first row, the first column, and the diagonal. All other terms may be neglected. With these simplifications, it is an easy matter to calculate $\tilde{\tilde{\rho}}$, $\tilde{\tilde{T}}$, and $\tilde{\tilde{u}}$, and we obtain

(IV.2)

$$\frac{\tilde{\tilde{\rho}}(k)}{L} = \tilde{\tilde{\sigma}}_1(k)(1+a_{11}) - a_{12}\tilde{\tilde{\sigma}}_2(k) - a_{13}\tilde{\tilde{\sigma}}_3^{(1)}(k) - a_{14}\tilde{\tilde{\sigma}}_3^{(2)}(k) - a_{15}\tilde{\tilde{\sigma}}_3^{(3)}(k)$$

$$\frac{\tilde{\tilde{T}}(k)}{L} = \tilde{\tilde{\sigma}}_2(k)(1+a_{22}) - a_{21}\tilde{\tilde{\sigma}}_2(k) - a_{23}\tilde{\tilde{\sigma}}_3^{(1)}(k) - a_{24}\tilde{\tilde{\sigma}}_3^{(2)}(k) - a_{25}\tilde{\tilde{\sigma}}_3^{(3)}(k)$$

$$\frac{u^{(1)}(k)}{L} = \tilde{\tilde{\sigma}}_3^{(1)}(k)(1+a_{33}) - a_{31}\tilde{\tilde{\sigma}}_1(k) - a_{32}\tilde{\tilde{\sigma}}_2(k) - a_{34}\tilde{\tilde{\sigma}}_3(k) - a_{35}\tilde{\tilde{\sigma}}_3^{(3)}(k)$$

$$\frac{u^{(2)}(k)}{L} = \tilde{\tilde{\sigma}}_3^{(2)}(k)(1+a_{44}) - a_{41}\tilde{\tilde{\sigma}}_1(k) - a_{42}\tilde{\tilde{\sigma}}_2(k) - a_{43}\tilde{\tilde{\sigma}}_3^{(1)}(k) - a_{45}\tilde{\tilde{\sigma}}_3^{(3)}(k)$$

$$\frac{u^{(3)}(k)}{L} = \tilde{\tilde{\sigma}}_3^{(3)}(k)(1+a_{55}) - a_{51}\tilde{\tilde{\sigma}}_1(k) - a_{52}\tilde{\tilde{\sigma}}_2(k) - a_{53}\tilde{\tilde{\sigma}}_3^{(1)}(k) - a_{54}\tilde{\tilde{\sigma}}_3^{(3)}(k)$$

We note that all the a_{ij} and all the source terms contain a factor $1/L$. Therefore, to determine the moments up to the necessary order as functions of x, it will be sufficient merely to invert the terms containing the source functions alone.

V. Discussion of the Source Term

For $L \longrightarrow \infty$, the source term $\tilde{\sigma}(\tilde{\xi})$ is precisely the free flow perturbation solution. This means that $\tilde{\sigma}(\tilde{\xi})$ at any point $P(x)$ is the difference between the stream of particles reflected by the body and the particles that are unable to reach P due to the presence of the body.

To $O(a^2/R^2)$, the source term may be expressed as

$$\tilde{\sigma}(\tilde{\xi}) = - \int_{\Sigma} \tilde{g}\, \tilde{\xi} \cdot dS \tag{V.1}$$

where the integral is taken over the surface of the body, the positive normal being directed inward. We refer to the article by Sirovich [14] for a derivation of (V.1).

It is apparent from (V.1) that the source term depends on the shape of the body and on the boundary conditions on $\tilde{g}(x,\tilde{\xi})$ both at the body and at infinity.

As a first approximation we assume $\tilde{g}(x,\tilde{\xi})$ to be the free flow perturbation solution.*

The unperturbed distribution function at ∞, as already mentioned, is Maxwellian:

$$\omega(\tilde{\xi}) = \frac{1}{(2\pi)^{3/2}} \exp\left\{ -\frac{1}{2} (\tilde{\xi} - \tilde{u}_o)^2 \right\} .$$

We consider the following cases:

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* We note here that this approximation will be improved upon by using an iteration procedure which we shall describe in detail in a later section.

a) Plate - specular reflection

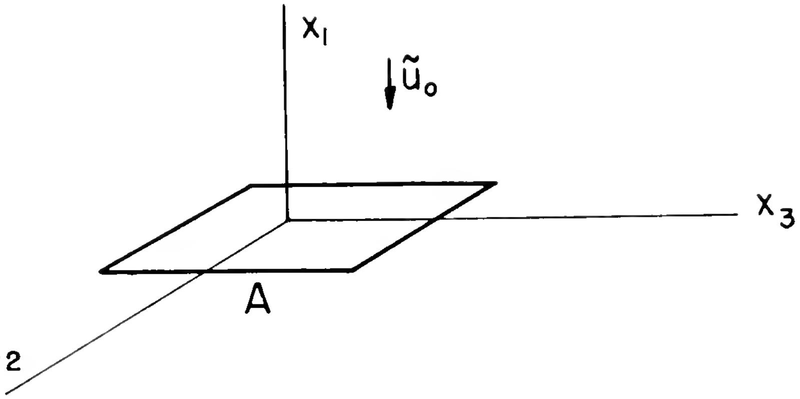

Figure 1

For a plate of area A moving in a direction x_1 normal to its plane we have (see Fig. 1)

$$\text{(V.2)} \quad \tilde{\sigma}(\tilde{\xi}) = \frac{A}{(2\pi)^{3/2}} \Big\{ |\tilde{\xi}_1| e^{-\frac{1}{2}[(\tilde{\xi}_1-\tilde{u}_o)^2+\tilde{\chi}^2]} H(\tilde{\xi}_1) - |\tilde{\xi}_1| e^{-\frac{1}{2}[(\tilde{\xi}_1+\tilde{u}_o)^2+\tilde{\chi}^2]} H(\tilde{\xi}_1) + |\tilde{\xi}_1| e^{-\frac{1}{2}[(\tilde{\xi}_1-\tilde{u}_o)^2+\tilde{\chi}^2]} H(-\tilde{\xi}_1) - |\tilde{\xi}_1| e^{-\frac{1}{2}[(\tilde{\xi}_1+\tilde{u}_o)^2+\tilde{\chi}^2]} H(-\tilde{\xi}_1) \Big\},$$

where

$$H(\tilde{\xi}_1) = 1 \quad \text{when} \quad \tilde{\xi}_1 > 0$$
$$= 0 \quad \text{when} \quad \tilde{\xi}_1 < 0$$

and

$$\tilde{\chi}^2 = \tilde{\xi}_2^2 + \tilde{\xi}_3^2 .$$

In expression (V.2), the first term represents the distribution of particles reflected from the front of the object;

from this we must subtract the distribution of particles incident on the back which is represented by the second term. The first two terms, therefore, account for the perturbed distribution at the front. Similarly, the next two terms represent the perturbed distribution at the back.

b) Plate - diffuse reflection

The source term is

(V.3) $$\tilde{\sigma}(\tilde{\xi}) =$$

$$= A\left\{ N_f^{(o)}|\tilde{\xi}_1|e^{-\frac{1}{2}\gamma\tilde{\xi}^2}H(\tilde{\xi}_1) - \frac{1}{(2\pi)^{3/2}}|\tilde{\xi}_1|e^{-\frac{1}{2}[(\tilde{\xi}_1+\tilde{u}_o)^2+\tilde{\chi}^2]}H(\tilde{\xi}_1)\right.$$

$$\left. + N_b^{(o)}|\tilde{\xi}_1|^{-\frac{1}{2}\gamma\tilde{\xi}^2}H(-\tilde{\xi}) - \frac{1}{(2\pi)^{3/2}}|\tilde{\xi}_1|e^{-\frac{1}{2}[(\tilde{\xi}_1+\tilde{u}_o)^2+\tilde{\chi}^2]}H(-\tilde{\xi}_1)\right\}$$

where $\gamma = T_i/T_r$, the ratio of the temperatures of the incident and reflected streams, and

(V.4) $$N_f^{(o)} =$$

$$\frac{\gamma^2}{(2\pi)^{3/2}}\left\{\sqrt{\frac{\pi}{2}}\,\tilde{u}_o\cos\theta + e^{-\frac{1}{2}\tilde{u}_o\cos\theta} + \sqrt{\frac{\pi}{2}}\,\tilde{u}_o\cos\theta\,\mathrm{erf}(\tfrac{1}{2}\tilde{u}_o\cos\theta)\right\}$$

$$N_b^{(o)} =$$

$$\frac{\gamma^2}{(2\pi)^{3/2}}\left\{-\sqrt{\frac{\pi}{2}}\,\tilde{u}_o\cos\theta + e^{-\frac{1}{2}\tilde{u}_o^2\cos^2\theta} + \sqrt{\frac{\pi}{2}}\,\tilde{u}_o\cos\theta\,\mathrm{erf}(\tfrac{1}{2}\tilde{u}_o\cos\theta)\right\}$$

with

$$\mathrm{erf}(\tfrac{1}{2}\tilde{u}_o) = \frac{2}{\sqrt{\pi}}\int_o^{\frac{1}{2}\tilde{u}_o} e^{-t^2}\,dt \ ,$$

c) <u>Sphere - specular reflection</u>

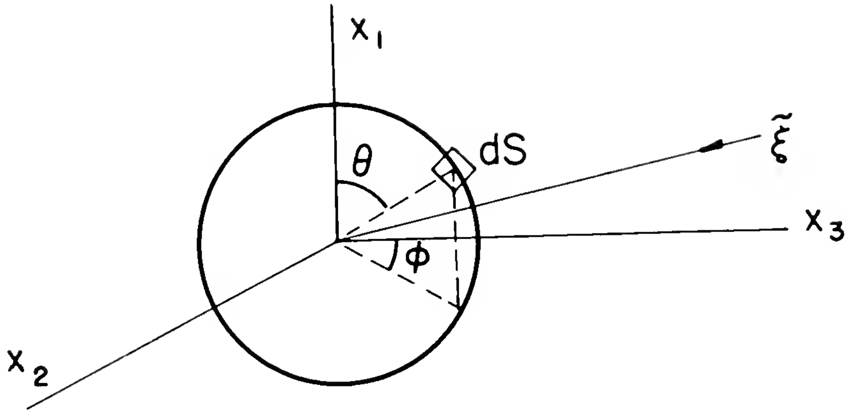

Figure 2

It is easily shown that the source term is given by

$$\text{(V.5)} \qquad \tilde{\sigma}(\tilde{\xi}) = \frac{\pi a^2}{(2\pi)^{3/2}} \Big\{ |\tilde{\xi}_1| e^{-\frac{1}{2}[(\tilde{\xi}_1 - \tilde{u}_o)^2 + \tilde{\chi}^2]} H(\tilde{\xi}_1)$$

$$- |\tilde{\xi}_1| e^{-\frac{1}{2}[(\tilde{\xi}_1 + \tilde{u}_o)^2 + \tilde{\chi}^2]} H(\tilde{\xi}_1) + |\tilde{\xi}_1| e^{-\frac{1}{2}[(\tilde{\xi}_1 - \tilde{u}_o)^2 + \tilde{\chi}^2]} H(-\tilde{\xi}_1)$$

$$- |\tilde{\xi}_1| e^{-\frac{1}{2}[(\tilde{\xi}_1 + \tilde{u}_o)^2 + \tilde{\chi}^2]} H(-\tilde{\xi}_1) \Big\}$$

where a = radius of the sphere.

d) <u>Sphere - diffuse reflection</u>

$$\text{(V.6)} \qquad \tilde{\sigma}(\tilde{\xi}) = \pi a^2 \Big\{ N_f^{(o)} |\tilde{\xi}_1| e^{-\frac{1}{2}\gamma\tilde{\xi}^2} H(\tilde{\xi}_1)$$

$$- \frac{1}{(2\pi)^{3/2}} |\tilde{\xi}_1| e^{-\frac{1}{2}[(\tilde{\xi}_1 + \tilde{u}_o)^2 + \tilde{\chi}^2]} H(\tilde{\xi}_1) + N_b^{(o)} |\tilde{\xi}_1| e^{-\frac{1}{2}\gamma\tilde{\xi}^2} H(-\tilde{\xi}_1)$$

$$- \frac{1}{(2\pi)^{3/2}} |\tilde{\xi}_1| e^{-\frac{1}{2}[(\tilde{\xi}_1 + \tilde{u}_o)^2 + \tilde{\chi}^2]} H(-\tilde{\xi}_1) \Big\} .$$

It is easy to check that in all cases the condition

$$\int \tilde{\sigma}(\tilde{\xi})d\tilde{\xi} = 0$$

is valid, i.e., conservation of number is satisfied.

VI. Evaluation of the Coefficients of the Matrix (III.11)

A knowledge of the source term $\tilde{\sigma}(\tilde{\xi})$, as discussed in the preceding section, enables us to calculate the terms $\bar{\bar{\sigma}}_1(k)$, $\bar{\bar{\sigma}}_2(k)$, $\bar{\bar{\sigma}}_3^i(k)$ of expression (III.11). After evaluation of the a_{ij}, the matrix is inverted and the expressions obtained for $\bar{\bar{\rho}}(k)$, $\bar{\bar{T}}(k)$ and $\bar{\bar{u}}(k)$, Fourier-transformed (Section VII). In Section VIII we shall solve equation (III.8) (i.e., the linearized Krook equation for the perturbed distribution function) where the inhomogeneous term is now a known function of x.

Although we shall not need the $a_{ij}(k)$ as such for our computation of the drag, we shall yet list them in Appendix I where they will serve for calculating the source terms.

We shall now describe how to evaluate a typical $a_{ij}(k)$ by integrating over $\tilde{\xi}$. The source terms may be integrated over $\tilde{\xi}$ in similar fashion as they are merely combinations of the components of the a_{ij}. We have, for example

$$\text{(VI.1)} \quad a_{11} = \int \Omega \, d\tilde{\xi} = \frac{1}{(2\pi)^{3/2}} \int \frac{\exp - \frac{1}{2}[(\tilde{\xi}_1 - \tilde{u}_0)^2 + \tilde{\xi}_2^2 + \tilde{\xi}_3]}{1 + iL\tilde{\xi}\cdot k} \, d\tilde{\xi} \ ,$$

assuming that the free stream velocity at ∞ is in the x_1 direction.

First, the k coordinate system is rotated so that $\vec{k}$ only has an x_1 component; hence $(\tilde{c}_1 + \tilde{u}_o)\cdot k = \tilde{c}_1'|k| + \tilde{u}_o\cdot k$, the primes referring to the new coordinate system (in the following calculation we shall drop the primes). Then

$$\text{(VI.2)} \qquad a_{11} = \frac{1}{\sqrt{2\pi}} \frac{1}{iL|k|} \int \frac{e^{-\frac{1}{2}\tilde{c}_1^2}}{(\tilde{c}_1 + \frac{1}{iL|k|} + \frac{\tilde{u}_o\cdot k}{|k|})} d\tilde{c}_1 \quad .$$

Expression (V.2) is of the form

$$\text{(VI.3)} \qquad I_o(z) = \int_{-\infty}^{\infty} \frac{e^{-t^2}dt}{(t-z)} \qquad \text{where} \quad z = x + iy,$$

which, as we shall show, may be reduced to a function of a tabulated integral [15].

Let $t - z = -u$, and assume $y > 0$; then $I_o(z) = e^{-z^2} f(z)$ where

$$f(z) = - \int_L \frac{e^{-u^2} e^{-2zu}}{u} du \ ,$$

the path of integration L being shown in Figure 3.

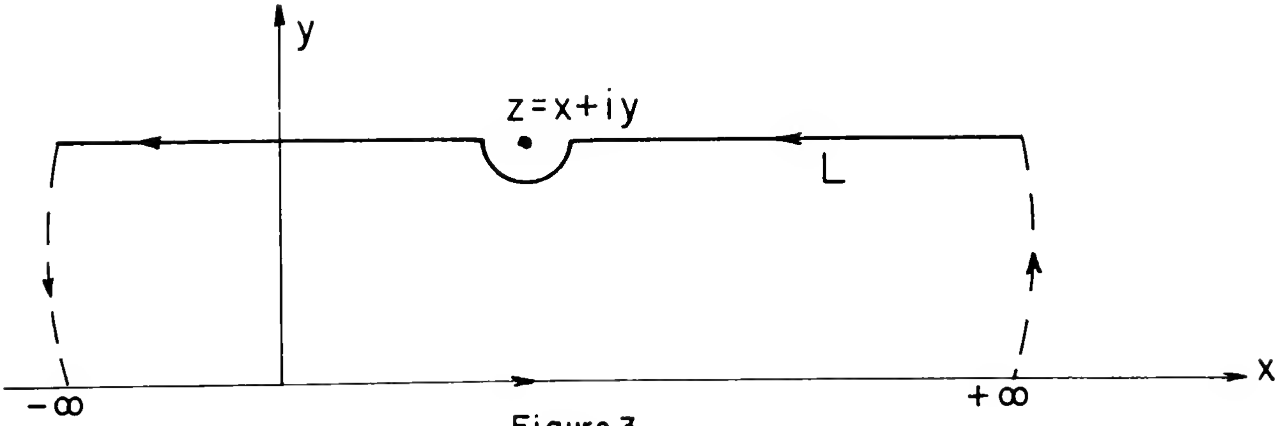

Figure 3

Now

$$\text{(VI.4)} \qquad f'(z) = -2\sqrt{\pi}\, e^{z^2}$$

and

$$\text{(VI.5)} \qquad f(z) = -2\sqrt{\pi} \int_0^z e^{t^2} dt + C \;,$$

where C is a constant to be determined. But

$$\lim_{y \to \infty} I_o(y) = \lim_{y \to \infty} \int_{-\infty}^{\infty} \frac{e^{-t^2} dt}{t - iy} = 0 \;.$$

Therefore, since $I_o(z) \to 0$ as $y \to \infty$, $f(z) = I_o(y) e^{-y^2}$ when $y \to \infty$ must also approach zero.

It follows that

$$\text{(VI.6)} \qquad C = \lim_{y \to \infty} 2\sqrt{\pi} \int_0^{yi} e^{t^2} dt = 2\sqrt{\pi}\, i \int_0^{\infty} e^{-u^2} du = \pi i \;,$$

(for $y < 0$ we would obtain $-\pi i$). Therefore,

$$\text{(VI.7)} \qquad I_o^{(1)}(z) = e^{-z^2} \left\{ \pi i - 2\sqrt{\pi} \int_0^z e^{t^2} dt \right\} \quad y > 0$$

$$\text{(VI.8)} \qquad I_o^{(2)}(z) = e^{-z^2} \left\{ -\pi i - 2\sqrt{\pi} \int_0^z e^{t^2} dt \right\} \quad y < 0$$

The exponential integral $\int_0^z e^{t^2} dt$ is tabulated in reference 15.

In terms of cur notation, we have

$$(\text{VI.9}) \qquad a_{11} = \frac{1}{\sqrt{\pi}\, iL|k|}\, e^{-\alpha^2} \left\{ \pi i - 2\sqrt{\pi}\, I(\alpha) \right\}$$

where

$$\alpha = \frac{1}{\sqrt{2}} \left(\frac{i}{L|k|} - \frac{\tilde{u}_o k_1}{|k|} \right)$$

with a positive imaginary part $1/L|k|$ and $I(\alpha) = \int_0^\alpha e^{t^2} dt$. The remaining coefficients a_{ij} of the matrix, together with an outline of their evaluation, are given in Appendix I.

For specular reflection, the source terms (III.11) integrate to

$$(\text{VI.10}) \qquad \tilde{\tilde{\sigma}}_1(k) = \frac{k_1 A}{iL|k|^2}\, e^{-\alpha^2} \left\{ i\sqrt{\pi}\, \alpha + e^{\alpha^2} - 2\alpha I(\alpha) \right\}$$
$$+ \frac{\tilde{u}_o A}{\sqrt{\pi}\, iL|k|}\, e^{-\alpha^2} \left\{ \pi i - 2\sqrt{\pi}\, I(\alpha) \right\} - \frac{k_1 A}{iL|k|^2}\, e^{-\beta^2} \left\{ i\sqrt{\pi}\, \beta + e^{\beta^2} - 2\beta I(\beta) \right\}$$
$$+ \frac{\tilde{u}_o A}{\sqrt{\pi}\, iL|k|}\, e^{-\beta^2} \left\{ \pi i - 2\sqrt{\pi}\, i(\beta) \right\} \equiv \tilde{\tilde{\sigma}}_1^{(r)}(\alpha,k) - \tilde{\tilde{\sigma}}_1^{(i)}(\beta,k)$$

$$(\text{VI.11}) \qquad \tilde{\tilde{\sigma}}_3^{(1)}(k) \equiv \frac{Ae^{-\alpha^2}}{iL|k|^3} \left\{ \frac{(k_2^2 + k_3^2)\sqrt{\pi}\, i}{2} + \sqrt{\pi}\, i\, k_1^2 \alpha^2 + k_1^2 \alpha e^{\alpha^2} \right.$$
$$\left. - 2I(\alpha) \left[\frac{(k_2^2 + k_3^2)}{2} + k_1^2 \alpha^2 \right] \right\} + \frac{A\tilde{u}_o k_1}{iL|k|^2}\, e^{-\alpha^2} \left\{ i\sqrt{\pi}\, \alpha + e^{\alpha^2} - 2I\alpha(\alpha) \right\}$$
$$- \frac{A}{iL|k|^2}\, e^{-\beta^2} \left\{ \frac{(k_2^2 + k_3^2)\sqrt{\pi}\, i}{2} + \sqrt{\pi}\, i\, k_1^2 \beta^2 + k_1^2 \beta e^{\beta^2} \right.$$
$$\left. - 2I(\beta) \left[\frac{(k_2^2 + k_3^2)}{2} + k_1^2 \beta^2 \right] \right\} + \frac{A\tilde{u}_o k_1}{iL|k|^2}\, e^{-\beta^2} \left\{ i\sqrt{\pi}\, \beta + e^{\beta^2} - 2\beta I(\beta) \right\}$$
$$\equiv \tilde{\tilde{\sigma}}_3^{(1)(r)}(\alpha,k) - \tilde{\tilde{\sigma}}_3^{(1)(i)}(\beta,k)$$

$$\text{(VI.12)} \quad \tilde{\tilde{\sigma}}_3^{(2)}(k) = \frac{Ak_1k_2}{iL|k|^3}\, e^{-\alpha^2}\left\{-\frac{\sqrt{\pi}\, i}{2} + \sqrt{\pi}\, i\alpha^2 + \alpha e^{\alpha^2} + I(\alpha)(1-2\alpha^2)\right\}$$

$$+ \frac{\tilde{u}_o A k_2 e^{-\alpha^2}}{iL|k|^2}\left\{i\sqrt{\pi}\,\alpha + e^{\alpha^2} - 2\alpha I(\alpha)\right\} - \frac{Ak_1k_2}{iL|k|^3}\, e^{-\beta^2}\left\{-\frac{\sqrt{\pi}\, i}{2}\right.$$

$$\left. + \sqrt{\pi}\, i\beta^2 + \beta e^{\beta^2} + I(\beta)(1-2\beta^2)\right\} + \frac{\tilde{u}_o A k_2 e^{-\beta^2}}{iL|k|^2}\left\{i\sqrt{\pi}\,\beta + e^{\beta^2}\right.$$

$$\left. - 2\beta I(\beta)\right\} \equiv \tilde{\tilde{\sigma}}_3^{(2)(r)}(\alpha,k) - \tilde{\tilde{\sigma}}_3^{(2)(i)}(\beta,k)$$

$\tilde{\tilde{\sigma}}_3^{(3)}(k)$ is the same as $\tilde{\tilde{\sigma}}_3^{(2)}(k)$ with k_2 replaced by k_3.

$$\text{(VI.13)} \quad \tilde{\tilde{\sigma}}_2(k) = \frac{1}{3}\,\frac{k_1 e^{-\alpha^2} A}{iL|k|^2}\left\{-2\sqrt{\pi}\, i\alpha + \sqrt{\pi}\, i\alpha^3 + e^{\alpha^2}(\alpha^2 - \tfrac{3}{2})\right.$$

$$\left. + 2\alpha I(\alpha)(2-\alpha^2)\right\} + \frac{\tilde{u}_o}{3}\,\frac{e^{-\alpha^2} A}{iL|k|}\left\{-2\sqrt{\pi}\, i + \sqrt{\pi}\, i\alpha^2 + \alpha e^{\alpha^2}\right.$$

$$\left. + 2(2-\alpha^2)I(\alpha)\right\} - \frac{1}{3}\, k_1\, \frac{e^{-\beta^2} A}{iL|k|^2}\left\{-2\sqrt{\pi}\, i\beta + \sqrt{\pi}\, i\beta^3 + e^{\beta^2}(\beta^2 - \tfrac{3}{2})\right.$$

$$\left. + 2\beta I(\beta)(2-\beta^2)\right\} + \frac{\tilde{\tilde{u}}_o}{3}\,\frac{e^{-\beta^2} A}{iL|k|}\left\{-2\sqrt{\pi}\, i + \sqrt{\pi}\, i\beta^2 + \beta e^{\beta^2}\right.$$

$$\left. + 2(2-\beta^2)I(\beta)\right\} \equiv \tilde{\tilde{\sigma}}_2^{r}(\alpha,k) - \tilde{\tilde{\sigma}}_2^{i}(\beta,k)$$

where

$$\alpha = \frac{1}{\sqrt{2}}\left\{\frac{i}{L|k|} - \frac{\tilde{u}_o k_1}{|k|}\right\}$$

$$\beta = \frac{1}{\sqrt{2}}\left\{\frac{i}{L|k|} + \frac{\tilde{u}_o k_1}{|k|}\right\} .$$

$\underline{A}$ may represent the area of the plate or the equatorial cross-section of the sphere. The meaning of the functions $\bar{\tilde{\sigma}}^{r}(\alpha,k)$ and $\bar{\tilde{\sigma}}^{i}(\beta,k)$ is obvious; the superscripts i and r refer to the impinging and reflected streams respectively. For diffuse reflection these source functions become

$$\text{(VI.14)}\quad \bar{\tilde{\sigma}}_1(k) = (N_f^{(o)} + N_b^{(o)}) \frac{k_1 A}{|k|^2} \frac{e^{\gamma/2L^2|k|^2}}{iL} \left\{ e^{-\gamma/2L^2|k|^2} - \frac{\sqrt{\gamma\pi}}{2L|k|} - \frac{\sqrt{2\gamma}\, i}{L|k|} I\left(\frac{i}{\sqrt{2}\, L|k|}\right)\right\} - \bar{\tilde{\sigma}}_1^{(i)}(\beta,k)$$

$$\text{(VI.15)}\quad \bar{\tilde{\sigma}}_3^{(1)}(k) = A(N_f^{(o)} + N_b^{(o)}) \frac{e^{\gamma/2L^2|k|^2}}{iL|k|^3} \left\{ \frac{(k_2^2 + k_3^2)\sqrt{\pi}\, i}{2} - \frac{\sqrt{\pi}\, i\, \gamma k_1^2}{2L^2|k|^2} + \sqrt{\frac{\gamma}{2}} \frac{i}{L|k|} k_1^2\, e^{-\gamma^2/2L^2|k|^2} - \left[(k_2^2 + k_3^2) - \frac{\gamma k_1^2}{L^2|k|^2}\right] I\left(\sqrt{\frac{\gamma}{2}} \frac{i}{L|k|}\right)\right\} - \bar{\tilde{\sigma}}_3^{(1)(i)}(\beta,k)$$

$$\text{(VI.16)}\quad \bar{\tilde{\sigma}}_3^{(2)}(k) = \frac{Ak_1k_2}{iL|k|^3} (N_f^{(o)} + N_b^{(o)})\, e^{\gamma/2L|k|^2} \left\{ -\frac{\sqrt{\pi}\, i}{2} - \frac{\sqrt{\pi}\, i\, \gamma}{2L^2|k|^2} + \sqrt{\frac{\gamma}{2}} \frac{i}{L|k|} e^{-\gamma/2L^2|k|^2} + \left(1 + \frac{\sqrt{2\gamma}}{L^2|k|^2}\right) I\left(\sqrt{\frac{\gamma}{2}} \frac{i}{L|k|}\right)\right\} - \bar{\tilde{\sigma}}_3^{(2)(i)}(\beta,k)$$

(VI.17) $\bar{\tilde{\sigma}}_3^{(3)}(k)$ is the same as $\bar{\tilde{\sigma}}_3^{(2)}(k)$ with k_2 replaced by k_3 .

$$(\mathrm{VI}.18) \quad \tilde{\tilde{\sigma}}_2(k) = \frac{1}{3}\left(N_f^{(o)} + N_b^{(o)}\right) \frac{k_1 A \; e^{\gamma/2L|k|^2}}{iL|k|^2} \left\{ \frac{\sqrt{2\gamma}}{L|k|} + \frac{\sqrt{\pi}}{L^3|k|^3}\left(\frac{\gamma}{2}\right)^{3/2} \right.$$

$$\left. - \; e^{-\gamma/2L|k|^2}\left(\frac{\gamma}{2L^2|k|^2} + \frac{3}{2}\right) + 2\sqrt{\frac{\gamma}{2}}\,\frac{i}{L|k|}\; I\left(\sqrt{\frac{\gamma}{2}}\,\frac{i}{L|k|}\right)\left(2 + \frac{\gamma}{2L^2|k|^2}\right)\right\}$$

$$- \; \tilde{\tilde{\sigma}}_2^{(i)}(\beta, k) \; .$$

For the purpose of calculating the drag, we shall only need to know these source terms up to $O(1/L)$. For specular reflection there is a considerable simplification in the source terms if this approximation is used and this case will be carried through for the plate. For diffuse reflection, however, the source terms are considerably more complicated.

Simplifications occur, however, if we let $\tilde{u}_o \longrightarrow \infty$, and results for a sphere undergoing diffuse reflection will be presented in this report.

VII. Evaluation of $\tilde{\rho}(x)$ for Specular Reflection

In this section, the Fourier inversion of $\tilde{\rho}(k)$ will be performed. We shall only consider specular reflection, reserving diffuse reflection for a subsequent section. The calculations will be performed using the free flow source terms discussed in Section V. The results obtained here will be valid for any Mach number and up to $O(1/L)$ in the Knudsen number.

We shall discuss the Fourier inversion of $\tilde{\bar{\rho}}(k)$ in some detail. The calculations for the other moments are essentially the same.

A typical term to be evaluated is

$$\text{(VII.1)} \qquad \tilde{\rho}_{oo}(x) = \frac{1}{(2\pi)^3}\int e^{ik\cdot x}\,\tilde{\bar{\sigma}}_1(k)dk =$$

$$\frac{8\pi\sqrt{\pi}\,A\tilde{u}_o}{(2\pi)^3 L}\int_0^\infty dk \int_0^\infty dk_1 \frac{e^{-\frac{u_o^2}{2}\frac{k_1^2}{k_1^2+\kappa^2}}}{(k_1^2+\kappa^2)^{1/2}}\; 1 - \frac{k_1^2}{\sqrt{2}\,(k_1^2+\kappa^2)}\quad \cos k_1 x_1 J_o(\kappa R)\kappa$$

where $R = \sqrt{x_2^2 + x_3^2}$, $\kappa = \sqrt{k_2^2 + k_3^2}$, and $J_o(x)$ is the Bessel function of zeroth order. The integrand of (VII.1) is now expanded in a series which may be shown to fulfill Abel's test

for uniform convergence [16].* Thus we may integrate the series term by term over k_1.

Intergration over k_1 yields

(VII.2) $\tilde{\rho}_{oo}(x) =$

$$\frac{8\pi^2 A\tilde{u}_o}{(2\pi)^3 L}\int_0^\infty d\mathscr{k}\, J_o(\mathscr{k} R)\sum_{n=o}^{\infty}\frac{(-1)^{3n}}{n!2^{2n}}\left(\frac{\tilde{u}_o}{\sqrt{2}}\right)^n x_1^{2n}\mathscr{k}^{2n+1}\left\{\frac{K_{2n}(\mathscr{k} x_1)}{\Gamma(2n+\frac{1}{2})}\right.$$

$$\left. - \frac{(-1)^{2n+1}}{2\sqrt{2}}\frac{\mathscr{k} x_1}{\Gamma(2n+\frac{3}{2})}K_{2n+1}(\mathscr{k} x_1)\right\}$$

using expression (11) of Appendix IV and properties (4), (7), and (8) of the modified Bessel functions $K_n(k)$, listed in Appendix III.

In order to integrate over $\mathscr{k}$ we must again show that the integrand is a uniformly convergent series in $\mathscr{k}$. Referring once more to Abel's test we have:

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* A discussion of Abel's (or Hardy's) test for uniform convergence is contained in [16]. It states essentially that if $v_n(x)$ be either monotonic increasing in n for each fixed x in (a,b) or monotonic decreasing in n for each fixed x in (a,b), then $\sum_n a_n(x)v_n(x)$ is uniformly convergent in (a,b) provided that

1) $\sum_n a_n(x)$ is uniformly convergent in (a,b),

2) $\exists\, k$ such that $|v_n(x)| < k$ for all n when $a \leq x \leq b$.

a) $$\sum_n \frac{1}{n!}\left(\frac{\tilde{u}_o^2}{2}\right)^n \frac{(-1)^{3n+1}}{2^{2n+1}\,\Gamma(2n+\frac{3}{2})}$$ and

$$\sum \frac{1}{n!}\left(\frac{\tilde{u}_o^2}{2}\right)^n \frac{(-1)^{3n}}{2^{2n}\,\Gamma(2n+\frac{1}{2})}$$

are uniformly convergent series independent of x_1.

b) The $K_n(k x_1)$ are monotonically increasing in n for a given $k x_1$.

c) $k^{2n+1} K_{2n}(k x_1)$ and $k^{2n+2} K_{2n+1}(k x_1)$ approach zero as k approaches zero from properties (1) and (2) of Appendix III.

d) $\lim_{k \to \infty} K_n(k x_1) \sim e^{-k}$ which approaches zero faster than k raised to any power.

We conclude that all the requirements of the test are satisfied and, therefore, that we may integrate (VII.2) term by term. Using expression (12) of Appendix IV we obtain (up to $O(1/L)$)

(VII.3) $\tilde{\rho}_{oo}(x) =$

$$\frac{A\tilde{u}_o}{\pi}\frac{1}{x_1^2}\sum_{n=o}^{\infty}\frac{(-1)^{3n}}{n!}\left(\frac{\tilde{u}_o}{\sqrt{2}}\right)^n\left\{\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}F(n+1,1;1;-\frac{R^2}{x_1^2})\right.$$

$$\left.-\frac{1}{\sqrt{2}}(-1)^3\cdot\frac{\Gamma(n+2)}{\Gamma(n+\frac{3}{2})}F(n+2,1;1;-\frac{R^2}{x_1^2})\right\}$$

where $R = \sqrt{x_2^2 + x_3^2}$ and $x_1 = r\cos\theta$ (see Figure 2) with

$r = \sqrt{R^2 + x_1^2}$.

The function F(a,b; c; z) is the hypergeometric function, some of whose properties are listed in Appendix V.

The evaluation of the other terms in $\tilde{\rho}(x)$ is entirely similar. For example, we have

$$\text{(VII.4)} \quad \tilde{\rho}_{11}(x) = \frac{1}{(2\pi)^3} \int e^{ik\cdot x}\, \tilde{\tilde{\sigma}}_1(k) a_{11}(k) dk$$

$$= 2 \cdot \frac{4\pi\sqrt{\pi}}{(2\pi)^3} \sum^{\infty} \frac{(-1)^n\, \tilde{u}_o^{2n}}{n!} \Bigg\{ \pi(-1)^{3n} \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} F(1, n+\tfrac{1}{2}; 1; -\frac{R^2}{x_1^2})$$

$$- \frac{\pi}{\sqrt{2}} (-1)^{3n+3} \frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+2)} F(1, n+\tfrac{3}{2}; 1; -\frac{R^2}{x_1^2}) - 2\sqrt{\pi} \sum_{p=o}^{\infty} \left(\frac{\tilde{u}_o}{\sqrt{2}}\right)^{2p+1} \cdot$$

$$\frac{(-1)^{3n+3p+2}}{(2P+1)p!} \frac{\Gamma(n+p+1)}{\Gamma(n+p+\frac{3}{2})} F(1, n+p+1; 1; -\frac{R^2}{x_1^2})$$

$$+ \frac{1}{\sqrt{2}} \sum_{p=o}^{\infty} \left(\frac{\tilde{u}_o}{\sqrt{2}}\right)^{2p+1} \frac{(-1)^{3n+3p+5}}{(2p+1)p!} \frac{\Gamma(n+p+2)}{\Gamma(n+p+\frac{5}{2})} F(k, n+p+2; 1; -\frac{R^2}{x_1^2}) \Bigg\}$$

where A may refer either to the area of the plate or to the cross-sectional area of the sphere.

VIII. Evaluation of $\tilde{\rho}(x)$, $\tilde{T}(x)$, and $\tilde{u}(x)$ at High Mach Number and for Diffuse Reflection

The values at the a_{ij} and the source terms, for high Mach number, are listed in Appendix II. For our purposes, we shall only need the source terms up to $O(1/L)$. These are

$$\tilde{\tilde{\sigma}}_1(k) = \frac{AN_f^{(o)}k_1}{iL|k|^2}$$

$$\tilde{\tilde{\sigma}}_3^{(1)}(k) = \frac{AN_f^{(o)}}{L|k|^3}\frac{\sqrt{\pi}}{2}(k_2^2+k_3^2) + \frac{A\tilde{u}_o}{iL|k|^2}k_1\left(1-\frac{1}{\sqrt{2}}\frac{k_1^2}{|k|^2}\right)$$

$$\text{(VIII.1)}\qquad \tilde{\tilde{\sigma}}_3^{(2)}(k) = -\frac{\sqrt{\pi}}{2}A\frac{N_f^{(o)}k_1k_2}{L|k|^3} + \frac{A\tilde{u}_ok_2}{iL|k|^2}\left(1-\frac{1}{\sqrt{2}}\frac{k_1^2}{|k|^2}\right)$$

$$\tilde{\tilde{\sigma}}_3^{(3)}(k) = -\frac{\sqrt{\pi}}{2}AN_f^{(o)}\frac{k_1k_3}{L|k|^3} + \frac{A\tilde{u}_ok_3}{iL|k|^2}\left(1-\frac{1}{\sqrt{2}}\frac{k_1^2}{|k|^2}\right)$$

$$\tilde{\tilde{\sigma}}_2(k) = \frac{A\tilde{u}_o^2}{iL|k|}\left(\frac{1}{3\sqrt{2}}\frac{k_1}{|k|} - \frac{1}{6}\frac{k_1^3}{|k|^3}\right)$$

Applying the Fourier inversion and following the methods of Section VII, we obtain

$$\text{(VIII.2)}\qquad \tilde{\rho}_{oo}(x) = \frac{L}{(2\pi)^3}\int e^{ik\cdot x}\,\tilde{\tilde{\sigma}}_1(k)dk = \frac{CN_f^{(o)}}{x_1^2}\,\Gamma(\tfrac{3}{2})F(\tfrac{3}{2},1;1;-\frac{R^2}{x_1^2})$$

$$\text{(VIII.3)}\qquad \tilde{T}_{oo}(x) = \frac{L}{(2\pi)^3}\int e^{ik\cdot x}\,\tilde{\tilde{\sigma}}_2(k)dk =$$

$$\frac{C}{3x_1^2}\left\{\frac{\tilde{u}_o^2}{\sqrt{2}}\,\Gamma(\tfrac{3}{2})F(\tfrac{3}{2},\ 1;\ 1;-\frac{R^2}{x_1^2}) - \frac{\tilde{u}_o^2}{2}\frac{\Gamma(5/2)}{\Gamma(2)}F(\tfrac{5}{2},\ 1;\ 1;-\frac{R^2}{x_1^2})\right\}$$

$$\text{(VIII.4)} \qquad \tilde{u}_{oo}^{(1)}(x) = \frac{L}{(2\pi)^3}\int e^{ik\cdot x}\,\tilde{\bar{\sigma}}_3^{(1)}(k)dk$$

$$= \frac{C}{x_1^2}\left\{\tilde{u}_o\Gamma(\tfrac{3}{2})F(\tfrac{3}{2},1;1;-\frac{R^2}{x_1^2}) + \frac{\tilde{u}_o}{\sqrt{2}}\,\frac{\Gamma(5/2)}{\Gamma(2)}\,F(\tfrac{5}{2},1;1;-\frac{R^2}{x_1^2})\right.$$

$$\left.+\frac{\sqrt{\pi}}{2}\,N_f^{(o)}\Gamma(\tfrac{3}{2})F(\tfrac{3}{2},\ 2;\ 1;-\frac{R^2}{x_1^2})\right\}$$

$$\text{(VIII.5)} \qquad \tilde{u}_{oo}^{(2)}(x) = \frac{L}{(2\pi)^3}\int e^{ik\cdot x}\,\tilde{\bar{\sigma}}_3^{(2)}(k)dk$$

$$= -\frac{AN_f^{(o)}\pi}{(2\pi)^2}\,\frac{\Gamma(2)}{\Gamma(3/2)}\cos\phi\,\frac{R}{x_1^3}\,F(2,2;2;-\frac{R^2}{x_1^2})$$

$$+\frac{2A\tilde{u}_o}{(2\pi)^2}\sqrt{\frac{\pi}{2}}\cos\phi\,\frac{\Gamma(3)\Gamma(3/2)}{\Gamma(2)\Gamma(2)}\cdot\frac{R}{x_1^3}\,F(3,\tfrac{3}{2};2;-\frac{R^2}{x_1^2})$$

$$+2(1-\frac{1}{\sqrt{2}})A\tilde{u}_o\,\frac{\sqrt{\pi}}{(2\pi)^2}\cos\phi\,\Gamma(\tfrac{3}{2})\,\frac{R}{x_1^3}\,F(2,\tfrac{3}{2};2;-\frac{R^2}{x_1^2})$$

$$\text{(VIII.6)} \qquad \tilde{u}_{oo}^{(3)}(x) = \frac{L}{(2\pi)^3}\int e^{ik\cdot x}\,\tilde{\bar{\sigma}}_3^{(3)}(k)dk$$ which is the same as $\tilde{u}_{oo}^{(2)}(x)$ with $\cos\phi$ replaced by $\sin\phi$,

where

$$C = \frac{4\pi\sqrt{\pi}\,A}{(2\pi)^3}$$

$$N_f^{(o)} = \frac{\gamma^2}{2\pi}\,\tilde{u}_o\cos\theta \quad \text{when} \quad \tilde{u}_o \to \infty.$$

IX. Evaluation of $\tilde{g}(x)$, the Perturbed Distribution Function for a High Mach Number Sphere Undergoing Diffuse Reflection

We start from equation (III.8), namely,

$$\tilde{\xi} \cdot \frac{\partial \tilde{g}}{\partial x} + \frac{1}{L}\,\tilde{g} = \frac{\omega}{L}\left\{\tilde{\rho}(x) + \left(\frac{1}{2}\,\tilde{c}^2 - \frac{3}{2}\right) \cdot \tilde{T}(x) + \tilde{c}\cdot\tilde{u}(x)\right\} + \tilde{\sigma}(\tilde{\xi})\delta(x)$$

of which the right-hand side is now a known function of x. We shall need to know $\tilde{\rho}(x)$, $\tilde{T}(x)$, $\tilde{u}(x)$ only to lowest order, i.e., we shall use $\tilde{\rho}_{oo}(x)$, $\tilde{T}_{oo}(x)$, $\tilde{u}_{oo}(x)$ in which the terms in $1/L$ are neglected.

For $\tilde{\sigma}(\tilde{\xi})$ we use expression (V.6). Transforming to spherical coordinates [18]* r, θ, ϕ, we have

$$\text{(IX.1)} \qquad \tilde{\xi}_r \cdot \frac{\partial \tilde{g}}{\partial r} + \tilde{\xi}_\theta\,\frac{\partial \tilde{g}}{r\partial\theta} + \frac{1}{L}\,\tilde{g} = \frac{\omega}{L}\left\{\tilde{\rho} + \left(\frac{1}{2}\,\tilde{c}^2 - \frac{3}{2}\right)\tilde{T} + \tilde{c}\cdot\tilde{u}\right\}$$

$$+ \tilde{\sigma}(\tilde{\xi}_r, \tilde{\xi}_\theta)\,\frac{\delta(r-a)\delta(\theta)}{2\pi r^2 \sin\theta}$$

for $x_1 > 0$ (for $x_1 < 0$, $\delta(\theta)$ is replaced by $\delta(\theta-\pi)$ in the source term). Here $\tilde{\xi}_r = \sqrt{\tilde{\xi}_1^2 + \tilde{\xi}_2^2 + \tilde{\xi}_3^2} = \sqrt{\tilde{\xi}_1^2 + \tilde{\chi}^2}$ and $\tilde{\xi}_\theta = \tan^{-1}\tilde{\chi}/\tilde{\xi}_1$.

Let $F(\tilde{g},\tilde{\xi}_r,\tilde{\xi}_\theta,r,\theta) = 0$ be a solution of (IX.1). Then $\frac{\partial\tilde{g}}{\partial r} = -\frac{\partial F}{\partial r}\Big/\frac{\partial F}{\partial \tilde{g}}$ and $\frac{\partial\tilde{g}}{\partial\theta} = -\frac{\partial F}{\partial\theta}\Big/\frac{\partial F}{\partial\tilde{g}}$ and equation (IX.1) becomes

---

* The transformation of a multi-dimensional δ-function from Cartesian to curvilinear coordinates is discussed in [18], pp. 291-293.

$$(\mathrm{IX.2}) \qquad \tilde{\xi}_r \frac{\partial F}{\partial r} + \frac{\tilde{\xi}_\theta}{r}\frac{\partial F}{\partial \theta} + \left[\frac{\omega}{L} G(r,\theta,\xi) - \frac{\tilde{g}}{L} + \tilde{\sigma}(\xi)\frac{\delta(r-a)\delta(\theta)}{2\pi r^2 \sin\theta}\right]\frac{\partial F}{\partial \tilde{g}} = 0$$

where $G(r,\theta,\xi) \equiv \tilde{\rho} + (\frac{1}{2}\tilde{c}^2 - \frac{3}{2})\tilde{T} + \tilde{c}\cdot\tilde{u}$.

Applying the method of characteristics to this equation for F, we have

$$(\mathrm{IX.3}) \qquad \frac{dr}{\tilde{\xi}_r} = \frac{r d\theta}{\tilde{\xi}_\theta} = \frac{d\tilde{g}}{\frac{\omega}{L} G - \frac{\tilde{g}}{L} + \frac{\tilde{\sigma}(\tilde{\xi})\delta(r-a,\theta)}{2\pi r^2 \sin\theta}}$$

which yields

$$(\mathrm{IX.4}) \qquad \frac{d\tilde{g}}{dr} + \frac{\tilde{g}}{L\tilde{\xi}_r} = \frac{1}{\tilde{\xi}_r}\left[\frac{\omega}{L} G + \frac{\tilde{\sigma}(\tilde{\xi})\delta(r-a,\theta)}{2\pi r^2 \sin\theta}\right] .$$

The solution is

$$(\mathrm{IX.5}) \qquad \tilde{g} = \frac{e^{-r/L\tilde{\xi}_r}}{\tilde{\xi}_r}\int\left\{\frac{\omega}{L} G(r,\theta,\xi) + \frac{\tilde{\sigma}(\tilde{\xi})\delta(r-a,\theta)}{2\pi r^2 \sin\theta}\right\} e^{r/L\tilde{\xi}_r}\, dr$$

using the boundary condition at ∞, namely, $\tilde{g} = 0$ for $r \rightarrow \infty$.

Since we only need $\tilde{g}$ up to $O(1/L)$, we expand the exponentials and obtain

$$(\mathrm{IX.6}) \qquad \tilde{g} = \tilde{g}^{(o)} + \frac{1}{L}\tilde{g}^{(1)} = -\frac{\omega r}{L\tilde{\xi}_r} G(r,\theta,\tilde{\xi})$$
$$+ \frac{\tilde{\sigma}(\tilde{\xi}_r,\tilde{\xi}_\theta)}{2\pi \sin\theta}\delta(\theta)\left[\frac{1}{a^2\tilde{\xi}_r} + \frac{1}{L\tilde{\xi}_r^2 a} - \frac{r}{a^2 L\tilde{\xi}_r^2}\right]$$

where, in $G(r,\theta,\tilde{\xi})$, we use expressions (VIII.2) to (VIII.6).

The function $g^{(1)}(\theta,a,\tilde{\xi})$ describes the perturbed velocity distribution function at the surface of the sphere. Granted that the object is sufficiently small, it is the correct first order perturbation and takes into account the effect of all collisions without any approximations.

We may also regard $g^{(1)}(\theta,a,\tilde{\xi})$ as describing the distribution arising from a modified source situated at the center of the sphere of radius a; this source gives rise to the perturbation in the drag.

X. Drag on a Sphere at High Mach Number and for Diffuse Reflection

We find the drag by calculating the momentum, in the direction $\tilde{u}_o$, imparted to a small sphere of radius <u>a</u> surrounding the source. The momentum due to the impinging molecules may be found directly from $\tilde{g}$. That due to the reflected stream may be found by applying the appropriate boundary conditions on $\tilde{g}$ at the surface of the sphere.

For diffuse reflection we have already calculated N_b and N_f to lowest order for the free stream case (expressions (IV.4)). We now find N_b and N_f to $O(1/L)$ from

$$
\text{(X.1)} \quad \left\{ \begin{aligned}
N_f^{(1)} \int_0^\infty d\tilde{\xi}_1 \int_{-\infty}^\infty d\tilde{\xi}_2 \int_{-\infty}^\infty d\tilde{\xi}_3 \tilde{\xi}_1 \, e^{-\frac{1}{2}\gamma\tilde{\xi}_r^2} &= - \int_{-\infty}^0 d\tilde{\xi}_1 \int_{-\infty}^\infty d\tilde{\xi}_2 \int_{-\infty}^\infty d\tilde{\xi}_3 \tilde{\xi}_1 g^{(1)} \\
N_b^{(1)} \int_{-\infty}^0 d\tilde{\xi}_1 \int_{-\infty}^\infty d\tilde{\xi}_2 \int_{-\infty}^\infty d\tilde{\xi}_3 \tilde{\xi}_1 \, e^{-\frac{1}{2}\gamma\tilde{\xi}_r^2} &= - \int_0^\infty d\tilde{\xi}_1 \int_{-\infty}^\infty d\tilde{\xi}_2 \int_{-\infty}^\infty d\tilde{\xi}_3 \tilde{\xi}_1 g^{(1)}
\end{aligned} \right.
$$

where $g^{(1)}$ is the term of $O(1/L)$ in $\tilde{g}$.

For $\tilde{u}_o \to \infty$ we have

$$
\text{(X.2)} \quad \left\{ \begin{aligned}
N_b^{(1)} &= 0 \\
N_f^{(1)} &= \frac{\gamma^2}{2\pi L} \frac{A(\tilde{u}_c, \theta)}{r^2 \cos^2\theta}
\end{aligned} \right.
$$

where

$$
A(\tilde{u}_o, \theta) = \frac{4\pi\sqrt{\pi}\,\pi a^2}{(2\pi)^3} N_f^{(o)} \Gamma(\tfrac{3}{2}) \, F(\tfrac{3}{2}, 1; 1; -\tan^2\theta) \quad ,
$$

with

$$N_f^{(o)} = \frac{2\gamma^2}{(2\pi)^{3/2}} \sqrt{\frac{\pi}{2}}\, \tilde{u}_o \cos\theta \quad \text{for} \quad \tilde{u}_o \to \infty .$$

Thus, the reflected stream has the form $N_f^{(1)} e^{-\frac{1}{2}\gamma\tilde{\xi}_r}$ where $\gamma = T_r/T_i$ (see Section V).

The drag due to the impinging stream in the $\tilde{u}_o$ direction is

$$\text{(X.3)} \qquad D_i = -\int dS \int_{-\infty}^{0} d\tilde{\xi}_1 \int_{-\infty}^{\infty} d\tilde{\xi}_2 \int_{-\infty}^{\infty} d\tilde{\xi}_3 \tilde{\xi}_1 (\tilde{\xi}_1 \cos\theta$$

$$+ \tilde{\xi}_2 \sin\theta) g(a,\theta,\tilde{\xi}_1 + \tilde{u}_o \cos\theta,\ \tilde{\xi}_2 + \tilde{u}_o \sin\theta,\ \tilde{\xi}_3)$$

Here dS is any element of surface on the sphere whose outward drawn normal is in the positive x_1 direction. Because the flow is symmetrical about $\tilde{u}_o$, we may always rotate the coordinate system about this direction so that u_o lies in the x_1x_2 plane (or x_1x_3 plane). We note that in calculating the drag due to the impinging molecules we need only consider those hitting the front of the sphere when $\tilde{u}_o \to \infty$.

The drag due to the reflected stream is

$$\text{(X.4)} \qquad D_r = -\int dS \int_{0}^{\infty} d\tilde{\xi}_1 \int_{-\infty}^{\infty} d\tilde{\xi}_2 \int_{-\infty}^{\infty} d\tilde{\xi}_3 (N_f^{(o)}$$

$$+ N_f^{(1)}) e^{-\frac{1}{2}\gamma(\tilde{\xi}_1^2+\tilde{\xi}_2^2+\tilde{\xi}_3^2)} \tilde{\xi}_1 (\tilde{\xi}_1 \cos\theta + \tilde{\xi}_2 \sin\theta)$$

again, only for the front of the sphere. The total drag is then

$$D = D_i + D_r \ .$$

If we let $L \rightarrow \infty$, we recover precisely the free-molecular flow drag. For L finite, the calculation of the drag reduces to an integration over θ between the limits 0 and $\pi/2$.

The integration over $\tilde{\xi}$ yields

$$\text{(X.5)} \quad \begin{cases} D_r = \dfrac{\pi^{3/2}\sqrt{2}}{\gamma^{5/2}} \displaystyle\int dS\,(N_f^{(o)} + N_f^{(1)})\cos\theta \\ D_i = -\pi a^2 \tilde{u}_o^2 + \dfrac{a}{L}\,\gamma^2\,\dfrac{\pi a^2 \tilde{u}_o^3}{2\pi\sqrt{\pi}} \displaystyle\int_o^{\pi/2} \sin\theta\; F(\tfrac{3}{2},1;1;-\tan^2\theta)\,d\theta \end{cases}$$

and $N_f^{(1)}$ is given by (X.2).

Using expression (5) of Appendix V, we may evaluate the integral in D_i which becomes the dominant term for $\tilde{u}_o \rightarrow \infty$. The final expression for the drag is then

$$\text{(X.6)} \qquad D = -\pi a^2 \left[\tilde{u}_o^2 - \frac{a}{L} \cdot \frac{\gamma^2 \tilde{u}_o^3}{8\sqrt{2}} \right] \ .$$

XI. Evaluation of g(x), the Perturbed Distribution Function, for a Plate at any Mach Number and for Specular Reflection

Using the same method as for a sphere, we solve equation (III.8) for $\tilde{g}$ using polar coordinates x_1, θ, and R. In the case of the plate, we are interested in the various moments at $x_1 = 0$ for the purpose of calculating the drag. This leads to considerable simplifications in the expressions for the moments and will enable us to evaluate the drag for all Mach numbers.

Using expression (4) of Appendix V, we find that the contributing terms for the plate, assuming specular reflection, are the following (up to O(1/L)):

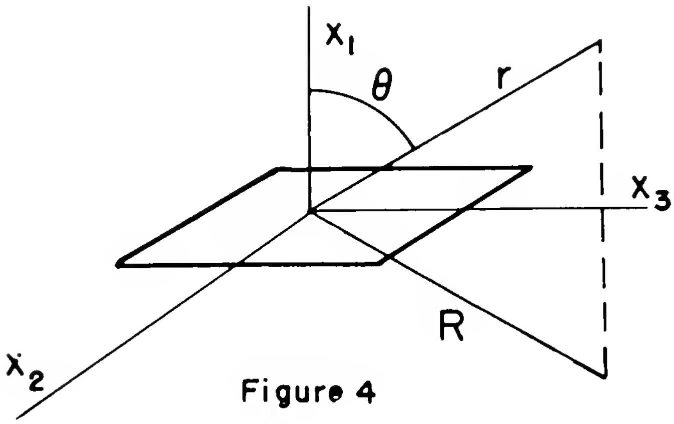

Figure 4

$$\tilde{\rho}(0,R) = \frac{16\pi\sqrt{\pi}\, A\tilde{u}_o}{(2\pi)^3 R^2}$$

$$\tilde{T}(0,R) = -\frac{32\pi^2 A\tilde{u}_o}{3(2\pi)^3 R^2} \qquad \text{(XI.1)}$$

$$\tilde{u}^{(1)}(0,R) = 0$$

for $x_1 = 0$ (i.e., in the plane of the plate).* In polar coordinates, the source term $\tilde{\sigma}(\tilde{\xi})\delta(x)$ becomes

$$\tilde{\sigma}(\tilde{\xi}_1, \tilde{\chi})\ \frac{\delta(x_1)\delta(R-a)}{2\pi R}\ .$$

* We do not need $\tilde{u}^{(2)}$ and $\tilde{u}^{(3)}$ to evaluate the drag in this case.

The solution is

$$\text{(XI.2)} \qquad \tilde{g}(\tilde{\xi}_1, \tilde{\chi}, R) = \frac{e^{-R/L\tilde{\chi}}}{\tilde{\chi}} \int \left\{ \frac{\omega}{L} G(0,R,\tilde{\xi}_1,\tilde{\chi}) + \frac{\tilde{\sigma}(\tilde{\xi}_1,\tilde{\chi})\delta(x_1)\delta(R-a)}{2\pi R} \right\} e^{R/L\tilde{\chi}} dR \ .$$

In order to eliminate the x_1 dependence in the source term (this x_1 dependence has already been eliminated in the term in G by using the values of $\tilde{\rho}$, $\tilde{T}$ and $\tilde{u}$ in the plane of the plate), we use the conditions on the characteristics, namely,

$$\frac{dx_1}{\xi_1} = \frac{dR}{\chi} \ .$$

Integrating the source term over x_1 yields*

$$\text{(XI.3)} \qquad \int \frac{\tilde{\sigma}(\tilde{\xi}_1,\tilde{\chi})\delta(R-a)}{2\pi R\tilde{\xi}_1} \delta(x_1)dx_1 = \frac{\tilde{\sigma}(\tilde{\xi}_1,\tilde{\chi})\delta(R-a)}{2\pi R\tilde{\xi}_1} \equiv g^{(o)}(\tilde{\xi}_1,\tilde{\chi},R)$$

and

$$\text{(XI.4)} \qquad \tilde{g}(\tilde{\xi}_1,\tilde{\chi},R) = \frac{\omega}{L\tilde{\chi}} \frac{A\tilde{u}_o}{(2\pi)^3 R} \left[-16\pi\sqrt{\pi} + \left(\frac{1}{2}\tilde{c}^2 - \frac{3}{2}\right) \frac{32\pi^2}{3} \right] + \frac{\tilde{\sigma}(\tilde{\xi}_1,\tilde{\chi})\delta(R-a)}{2\pi R\tilde{\xi}_1} \ .$$

The first order correction to the drag is

---

* It is easily seen that the drag due to $g^{(o)}(\tilde{\xi}_1, \tilde{\chi}, R)$ is precisely the free molecular flow drag.

$$(XI.5) \qquad D^{(1)} = - \frac{4A\tilde{u}_o}{(2\pi)^2 (2\pi)^{3/2} L} \left\{ \tilde{u}_o (k_1 + 12k_2) e^{-\frac{1}{2}\tilde{u}_o^2} + [k_1 (1 + \tilde{u}_o^2) + 2k_2] \int_0^{\tilde{u}_o} e^{-\frac{1}{2}x^2} dx - 2\tilde{u}_o (k_1 + 8k_2) \right\}$$

where

$$k_1 \equiv - 16\pi \sqrt{\pi}$$

$$k_2 \equiv \frac{32\pi^2}{3} \quad .$$

At high Mach number, the total drag approaches

$$(XI.6) \qquad D = -A\tilde{u}_o^2 + \frac{a}{L} \frac{16\sqrt{\pi}\ A}{(2\pi)^2} \tilde{u}_o^3 \ .$$

Here, as for the sphere, collisions between particles bring about a reduction in the drag.

Appendix I: Evaluation of $\displaystyle\int_{-\infty}^{\infty} \frac{t^n e^{-t^2} dt}{t-z}$ $\quad n = 1,2,3,4.$

The case of n = 0 has already been explained in the text. All cases with $n \geq 1$ may be reduced to the $n = 0$ case.

$$1) \qquad I_1(z) = \int_{-\infty}^{\infty} \frac{t\, e^{-t^2} dt}{t-z} = \int_{-\infty}^{\infty} \frac{(t-z)+z}{(t-z)}\, e^{-t^2} dt \qquad z = x+iy$$

yielding

$$(1.\mathrm{i}) \quad \begin{cases} I_1^{(1)}(z) = e^{-z^2}\sqrt{\pi}\left\{ i\sqrt{\pi}\, z + e^{z^2} - 2zI(z) \right\} & y > 0 \\ I_1^{(2)}(z) = e^{-z^2}\sqrt{\pi}\left\{ -i\sqrt{\pi}\, z + e^{z^2} - 2zI(z) \right\} & y < 0 \end{cases}$$

where

$$I(z) = \int_0^z e^{\beta^2} d\beta \; .$$

$$2) \qquad I_2(z) = \int_{-\infty}^{\infty} \frac{t^2 e^{-t^2} dt}{(t-z)} = \int_{-\infty}^{\infty} \frac{(t^2 - z^2) + z^2}{(t-z)}\, e^{t^2} dt$$

$$= \int_{-\infty}^{\infty} \frac{(t-z)(t+z) + z^2}{(t-z)}\, e^{-t^2} dt \; .$$

Thus

$$(1.\mathrm{ii}) \quad \begin{cases} I_2^{(1)}(z) = e^{-z^2}\sqrt{\pi}\left\{ i\sqrt{\pi}\, z^2 + ze^{-z^2} - 2z^2 I(z) \right\} & y > 0 \\ I_2^{(2)}(z) = e^{-z^2}\sqrt{\pi}\left\{ -i\sqrt{\pi}\, z^2 + ze^{-z^2} - 2z^2 I(z) \right\} & y < 0 \; . \end{cases}$$

3) $$I_3(z) = \int_{-\infty}^{\infty} \frac{t^3 e^{-t^2} dt}{(t-z)} = \int_{-\infty}^{\infty} \frac{(t-z+z)t^2}{(t-z)} e^{-t^2} dt$$

yielding

$$(1.\text{iii}) \quad \begin{cases} I_3^{(1)}(z) = \sqrt{\pi}\, e^{-z^2}\left\{\sqrt{\pi}\, iz^3 + \frac{e^{z^2}}{2}(1+2z^2) - 2z^3 I(z)\right\} & y > 0 \\ I_3^{(2)}(z) = \sqrt{\pi}\, e^{-z^2}\left\{-\sqrt{\pi}\, iz^3 + \frac{e^{z^2}}{2}(1+2z^2) - 2z^3 I(z)\right\} & y < 0 \end{cases}$$

4) $$I_4(z) = \int_{-\infty}^{\infty} \frac{t^4 e^{-t^2} dt}{(t-z)} = \int_{-\infty}^{\infty} \frac{(t-z+z)t^3}{(t-z)} e^{-t^2} dt$$

yielding

$$(1.\text{iv}) \quad \begin{cases} I_4^{(1)}(z) = \sqrt{\pi}\, z e^{-z^2}\left\{\sqrt{\pi}\, iz^3 + \frac{e^{z^2}}{2}(1+2z^2) - 2z^3 I(z)\right\} & y > 0 \\ I_4^{(2)}(z) = \sqrt{\pi}\, z e^{-z^2}\left\{-\sqrt{\pi}\, iz^3 + \frac{e^{z^2}}{2}(1+2z^2) - 2z^3 I(z)\right\} & y < 0 \end{cases}$$

Integrals with values of $n > 4$ are not needed in our calculations and are, therefore, not tabulated.

The remaining coefficients of the matrix (III.11) are

$$a_{12} = \frac{1}{2}\int (\tilde{c}^2 - 3)\Omega d\tilde{\xi}$$

$$= -\frac{1}{2iL|k|} e^{-\alpha^2}\left\{-2\sqrt{\pi}\, i + \sqrt{\pi}\, i\alpha^2 + \alpha e^{\alpha^2} + 2(2-\alpha^2)I(\alpha)\right\}$$

$$a_{21} = -\frac{1}{3}\int (\tilde{c}^2 - 3)\Omega d\tilde{\xi} = \frac{2}{3} a_{12} \qquad \alpha \equiv \frac{1}{\sqrt{2}}\left[\frac{i}{L|k|} - \frac{\tilde{u}_o k_1}{|k|}\right]$$

$$a_{13} = \int \tilde{c}_1 \Omega d\tilde{\xi} = -\frac{k_1}{iL|k|^2} e^{-\alpha^2}\left\{ i\sqrt{\pi}\,\alpha + e^{\alpha^2} - 2\alpha I(\alpha)\right\}$$

$$a_{31} = a_{13}$$

$$a_{14} = -\int \tilde{c}_2 \Omega d\tilde{\xi} = -\frac{k_2}{iL|k|^2} e^{-\alpha^2}\left\{ i\sqrt{\pi}\,\alpha + e^{\alpha^2} - 2\alpha I(\alpha)\right\}$$

$$a_{41} = a_{14}$$

$$a_{15} = -\int \tilde{c}_3 \Omega d\tilde{\xi} = -\frac{k_3}{iL|k|^2} e^{-\alpha^2}\left\{ i\sqrt{\pi}\,\alpha + e^{\alpha^2} - 2\alpha I(\alpha)\right\}$$

$$a_{51} = a_{15}$$

$$a_{22} = \frac{1}{6}\int (\tilde{c}^2 - 3)^2 \Omega d\tilde{\xi} = \frac{e^{-\alpha^2}}{6iL|k|}\left\{5\sqrt{\pi}\, i - 4\sqrt{\pi}\, i\alpha^2 + \sqrt{\pi}\, i\alpha^4 + e^{\alpha^2}[-\tfrac{7}{2}\alpha + \alpha^3] + 2I(\alpha)[-5 + 4\alpha^2 - \alpha^4]\right\}$$

$$a_{23} = -\frac{1}{3}\int \tilde{c}_1(\tilde{c}^2 - 3)\Omega d\tilde{\xi} = -\frac{1}{3}\frac{k_1 e^{-\alpha^2}}{iL|k|^2}\left\{-2\sqrt{\pi}\, i\alpha + \sqrt{\pi}\, i\alpha^3 + e^{\alpha^2}(-\tfrac{3}{2} + \alpha^2) + 2\alpha I(\alpha)[2-\alpha^2]\right\}$$

$$a_{32} = -\frac{1}{2}\int \tilde{c}_1(\tilde{c}^2 - 3)\Omega d\tilde{\xi} = \frac{3}{2}\, a_{23}$$

$$a_{24} = -\frac{1}{3}\int \tilde{c}_2(\tilde{c}^2 - 3)\Omega d\tilde{\xi} = -\frac{1}{3}\frac{k_2 e^{-\alpha^2}}{iL|k|^2}\left\{-2\sqrt{\pi}\, i\alpha + \sqrt{\pi}\, i\alpha^3 + e^{\alpha^2}(-\tfrac{3}{2} + \alpha^2) + 2\alpha I(\alpha)[2-\alpha^2]\right\}$$

$$a_{42} = -\frac{1}{2}\int \tilde{c}_2(\tilde{c}^2 - 3)\Omega d\tilde{\xi} = \frac{3}{2}\, a_{24}$$

$$a_{25} = -\frac{1}{3}\int \tilde{c}_3(\tilde{c}^2 - 3)\Omega d\tilde{\xi} = -\frac{1}{3}\,\frac{k_3 e^{-\alpha^2}}{iL|k|^2}\Big\{2\sqrt{\pi}\, i\alpha + \sqrt{\pi}\, i\alpha^3$$

$$+ e^{\alpha^2}\left(-\frac{3}{2} + \alpha^2\right) + 2\alpha I(\alpha)[2-\alpha^2]\Big\}$$

$$a_{52} = -\frac{1}{3}\int \tilde{c}_3(\tilde{c}^2 - 3)\Omega d\tilde{\xi} = \frac{3}{2}\, a_{25}$$

$$a_{33} = \int \tilde{c}_1^2\, \Omega\, d\tilde{\xi} = \frac{e^{-\alpha^2}}{iL|k|^3}\Big\{\frac{(k_2^2 + k_3^2)\sqrt{\pi}\, i}{2} + \sqrt{\pi}\, i\, k_1^2\alpha^2$$

$$+ k_1^2\alpha e^{\alpha^2} - 2I(\alpha)\left[\frac{(k_2^2 + k_3^2)}{2} + k_1^2\alpha^2\right]\Big\}$$

$$a_{34} = -\int \tilde{c}_1\tilde{c}_2\Omega d\tilde{\xi} = -\frac{k_1 k_2}{iL|k|^3}\, e^{-\alpha^2}\Big\{-\frac{\sqrt{\pi}\, i}{2} + \sqrt{\pi}\, i\alpha^2 + \alpha e^{\alpha^2}$$

$$+ I(\alpha)[1-2\alpha^2]\Big\}$$

$$a_{35} = -\int \tilde{c}_1\tilde{c}_3\Omega d\tilde{\xi} = -\frac{k_1 k_3}{iL|k|^3}\, e^{-\alpha^2}\Big\{-\frac{\sqrt{\pi}\, i}{2} + \sqrt{\pi}\, i\alpha^2 + \alpha e^2$$

$$+ I(\alpha)[1-2\alpha^2]\Big\}$$

$$a_{53} = -\int \tilde{c}_3\tilde{c}_1\Omega d\tilde{\xi} = a_{35}$$

$$a_{43} = -\int \tilde{c}_2\tilde{c}_1\Omega d\tilde{\xi} = a_{35}$$

$$a_{43} = -\int \tilde{c}_2\tilde{c}_1\Omega d\tilde{\xi} = a_{34}$$

$$a_{44} = \int \tilde{c}_2^2\, \Omega\, d\tilde{\xi} = \frac{e^{-\alpha^2}}{iL|k|}\left\{\frac{\sqrt{\pi}\, i}{2}\left[\frac{k_3^2}{(k_2^2+k_3^2)} + \frac{k_1 k_2^2}{(k_2^2+k_3^2)|k|^2}\right]\right.$$

$$+ \frac{k_2^2}{|k|^2}\sqrt{\pi}\, i\alpha^2 + \frac{k_2^2\, \alpha\, e^{\alpha^2}}{|k|^2} - 2I(\alpha)\left[\frac{k_1 k_2^2}{2(k_2^2+k_3^2)|k|^2} + \frac{k_2^2}{2(k_2^2+k_3^2)}\right.$$

$$\left.+ \frac{\alpha^2 k_2^2}{|k|^2}\right]$$

$$a_{45} = -\int \tilde{c}_2 \tilde{c}_3 \Omega d\tilde{\xi} = -\frac{e^{-\alpha^2}}{iL|k|}\left\{\frac{\sqrt{\pi}\, i}{2}\left[\frac{k_1^2 k_2 k_3}{(k_2^2+k_3^2)|k|^2} - \frac{k_2 k_3}{(k_2^2+k_3^2)}\right]\right.$$

$$+ \frac{2k_2k_3}{|k|^2}\sqrt{\pi}\, i\alpha^2 + \frac{k_2k_3\alpha e^{\alpha^2}}{LkL^2} + I(\alpha)\left[\frac{k_2k_3}{(k_2^2+k_3^2)} - \frac{k_1^2 k_2 k_3}{(k_2^2+k_3^2)|k|^2}\right.$$

$$\left.\left. - \frac{2k_2k_3\alpha^2}{|k|^2}\right]\right\}$$

$$a_{54} = \int \tilde{c}_3 \tilde{c}_2 \Omega d\tilde{\xi} = a_{45}$$

$$a_{55} = \int \tilde{c}_3^2\, \Omega\, d\tilde{\xi} = \frac{e^{-\alpha^2}}{iL|k|}\left\{\frac{\sqrt{\pi}\, i}{2}\left[\frac{k_2^2}{(k_2^2+k_3^2)} + \frac{k_1 k_3}{(k_2^2+k_3^2)|k|^2}\right]\right.$$

$$+ \frac{\sqrt{\pi}\, i k_3^2}{|k|^2}\alpha^2 + \frac{k_3^2}{|k|^2}\alpha e^{\alpha^2} - 2I(\alpha)\left[\frac{k_1^2 k_3^2}{2(k_2^2+k_3^2)|k|^2} + \frac{k_2^2}{2(k_2^2+k_3^2)}\right.$$

$$\left.\left. + \frac{k_3^2\alpha^2}{|k|^2}\right]\right\} .^{*}$$

The values of these coefficients at high Mach numbers are listed in Appendix II.

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* In calculating the a_{ij} we use the relations

$$\xi_1' = \frac{k_1}{|k|}\ \xi_1 + \frac{k_2}{|k|}\ \xi_2 + \frac{k_3}{|k|}\ \xi_3$$

$$\xi_2' = \frac{k_3}{\sqrt{k_2^2 + k_3^2}}\ \xi_2 - \frac{k_2}{\sqrt{k_2^2 + k_3^2}}\ \xi_3$$

$$\xi_3' = -\frac{\sqrt{k_2^2 + k_3^2}}{|k|}\ \xi_1 + \frac{k_1 k_2}{\sqrt{k_2^2 + k_3^2}}\ \xi_2 + \frac{k_3 k_1}{\sqrt{k_2^2 + k_3^2}\ |k|}\ \xi_3$$

expressing the new coordinates ξ_1', ξ_2', ξ_3' in terms of the old. The inverse relations are

$$\xi_1 = \frac{k_1}{|k|}\ \xi_1' - \frac{\sqrt{k_2^2 + k_3^2}}{|k|}\ \xi_3'$$

$$\xi_2 = \frac{k_2}{|k|}\ \xi_1' + \frac{k_3}{\sqrt{k_2^2 + k_3^2}}\ \xi_2' + \frac{k_1 k_2}{\sqrt{k_2^2 + k_3^2}\ |k|}\ \xi_3'$$

$$\xi_3 = \frac{k_3}{|k|}\ \xi_1' - \frac{k_2}{\sqrt{k_2^2 + k_3^2}}\ \xi_2' + \frac{k_3 k_1}{\sqrt{k_2^2 + k_3^2}\ |k|}\ \xi_3' \quad .$$

Appendix II:

(a) Values of the coefficients a_{ij} and the source terms $\tilde{\sigma}(k)$ when $\tilde{u}_o \to \infty$.

The leading terms to $O(1/L^2)$ are:

$$a_{11} \sim 0$$

$$a_{12} \sim \frac{-1}{2\sqrt{2}\, iL|k|} \left(\frac{i}{L|k|} - \frac{\tilde{u}_o k_1}{|k|} \right) \qquad a_{21} = \frac{2}{3} a_{12}$$

$$a_{13} \sim - \frac{k_1}{iL|k|^2} \qquad a_{31} = a_{13}$$

$$a_{14} \sim - \frac{k_2}{iL|k|^2} \qquad a_{41} = a_{14}$$

$$a_{15} \sim - \frac{k_3}{iL|k|^2} \qquad a_{51} = a_{15}$$

$$a_{22} \sim \frac{\tilde{u}_o^2 k_1^2}{12\sqrt{2}\, iL|k|^4} \left\{ i - \tilde{u}_o k_1 \right\}$$

$$a_{23} \sim - \frac{\tilde{u}_o k_1^2}{6iL|k|^4} \left\{ \tilde{u}_o k_1 - \frac{2i}{L} \right\} \qquad a_{32} = \frac{3}{2} a_{23}$$

$$a_{24} \sim - \frac{\tilde{u}_o k_2 k_1}{6iL|k|^4} \left\{ \tilde{u}_o k_1 - \frac{2i}{L} \right\} \qquad a_{42} = \frac{3}{2} a_{24}$$

$$a_{25} \sim - \frac{\tilde{u}_o k_3 k_1}{6iL|k|^4} \left\{ \tilde{u}_o k_1 - \frac{2i}{L} \right\} \qquad a_{52} = \frac{3}{2} a_{25}$$

$$a_{33} \sim \frac{k_1^3}{\sqrt{2}\, iL|k|^3} \left\{ - \frac{\tilde{u}_o k_1}{|k|} + \frac{i}{L|k|} \right\}$$

$$a_{34} \sim - \frac{k_1 k_2}{\sqrt{2}\, iL|k|^3} \left\{ - \frac{\tilde{u}_o k_1}{|k|} + \frac{i}{L|k|} \right\} \qquad a_{43} = a_{34}$$

$$\tilde{\tilde{\sigma}}_3^{(1)}(k) \sim \frac{AN_f^{(o)}}{L|k|^3}\left\{\frac{(k_2^2+k_3^2)\sqrt{\pi}}{2} + \sqrt{\frac{\gamma}{2}}\,\frac{k_1^2}{L|k|} - \frac{(k_2^2+k_3^2)}{L|k|}\sqrt{\frac{\gamma}{2}}\right\}$$

$$+ \frac{Ak_1\tilde{u}_o}{iL|k|^2}\left(1 - \frac{1}{\sqrt{2}}\,\frac{k_1^2}{|k|^2}\right)$$

$$\tilde{\tilde{\sigma}}_3^{(2)}(k) \sim \frac{Ak_1k_2N_f^{(o)}}{L|k|^3}\left\{-\frac{\sqrt{\pi}}{2} + \frac{\sqrt{2\gamma}}{L|k|}\right\} + \frac{Ak_2\tilde{u}_o}{iL|k|^2}\left(1 - \frac{1}{\sqrt{2}}\,\frac{k_1^2}{|k|^2}\right)$$

$$\tilde{\tilde{\sigma}}_3^{(3)}(k) \sim \frac{Ak_1k_3N_f^{(o)}}{L|k|^3}\left\{-\frac{\sqrt{\pi}}{2} + \frac{\sqrt{2\gamma}}{L|k|}\right\} + \frac{Ak_3\tilde{u}_o}{iL|k|^2}\left(1 - \frac{1}{\sqrt{2}}\,\frac{k_1^2}{|k|^2}\right)$$

$$\tilde{\tilde{\sigma}}_2(k) \sim \frac{\sqrt{2\gamma}\,N_f^{(o)}Ak_1}{3iL^2|k|^3} - \frac{1}{2}N_f^{(o)}\frac{Ak_1}{iL|k|^2} + \frac{A\tilde{u}_o^2}{iL|k|}\left(\frac{1}{3\sqrt{2}} - \frac{1}{6}\,\frac{k_1^3}{|k|^3}\right)$$

$$+ \frac{A\tilde{u}_o}{3L^2|k|^3}\left(\frac{1}{\sqrt{2}} - \frac{k_1^2}{|k|^2}\right)$$

for diffuse reflection.

(b) <u>Source Terms for any Mach Number and Specular Reflection</u>

$$\left(\alpha \equiv -\frac{1}{\sqrt{2}}\,\frac{\tilde{u}_o k_1}{|k|}\right)$$

$$\tilde{\tilde{\sigma}}_1(k) = \frac{2\sqrt{\pi}\,A}{L|k|}\,e^{-\alpha^2}\left\{\frac{\alpha k_1}{|k|} + \tilde{u}_o\right\}$$

$$\tilde{\tilde{\sigma}}_3^{(1)}(k) = \frac{2Ae^{-\alpha^2}}{iL}\left\{e^{\alpha^2}\left(\frac{k_1^2}{|k|^3}\alpha + \tilde{u}_o\frac{k_1}{|k|^2}\right) - 2\left[\frac{(k_2^2+k_3^2)}{2|k|^3}\right.\right.$$

$$\left.\left.+ \frac{k_1^2}{|k|^3}\alpha^2 - 2\alpha\frac{k_1}{|k|^2}\right]I(\alpha)\right\}$$

$$\tilde{\tilde{\sigma}}_3^{(2)}(k) = \frac{2Ak_2}{iL|k|^2}\,e^{-\alpha^2}\left\{\frac{k_1}{|k|}\left[\alpha e^{\alpha^2} + (1-2\alpha^2)I(\alpha)\right] + i\sqrt{\pi}\,\tilde{u}_o\alpha\right\}$$

$$\tilde{\tilde{\sigma}}_3^{(3)}(k) = \frac{2Ak_3}{iL|k|^3}\,e^{-\alpha^2}\left\{\frac{k_1}{|k|}\left[\alpha e^{\alpha^2} + (1-2\alpha^2)I(\alpha)\right] + i\sqrt{\pi}\,\tilde{u}_o\alpha\right\}$$

$$\tilde{\bar{\sigma}}_2(k) = \frac{2\sqrt{\pi}\, e^{-\alpha^2}}{3L|k|} \left\{ -2\tilde{u}_o - 2\frac{k_1}{|k|}\alpha + \tilde{u}_o\alpha^2 + \frac{k_1}{|k|}\alpha^3 \right\} .$$

Appendix III: Some Properties of the Modified Bessel Functions*

The modified Bessel functions of the first and second kinds, $I_\nu(z)$ and $K_\nu(z)$, are defined as solutions of the equation

$$z^2 \frac{d^2y}{dz^2} + z\frac{dy}{dz} - (z^2 + \nu^2)y = 0 .$$

We have $I_\nu(z) = e^{-\nu\pi i/2} J_\nu(iz)$ which is a real function of z, and $K_\nu(z) = \frac{\pi}{2} e^{(\nu+1)\pi i/2} H_\nu^{(1)}(iz)$.

For ν not an integer, we have

$$K_\nu(z) = \frac{1}{2}\pi \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi} .$$

For $0 < x < 1$

$$K_o(z) \approx \ell n \frac{2}{\gamma^2}$$

$$K_n(z) \approx \frac{1}{2}\Gamma(n)\left(\frac{2}{z}\right)^n \qquad n = 1,2,\ldots$$

(γ = Euler's constant).

Asymptotically,

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}}\, e^{-z} .$$

* For further properties of these functions see [19] and [20].

$K_\nu(z)$ exists for all finite values of ν and is monotonically increasing with ν for a given z. For all values of ν, however, $K_\nu(0) = \infty$.

The $K_\nu(z)$ satisfy the following relations:

1) $$K_{\nu-1}(z) - K_{\nu+1}(z) = -\frac{2\nu}{z} K_\nu(z)$$

2) $$K_{\nu-1}(z) + \nu K_\nu(z) = -2K'_\nu(z)$$

3) $$zK'_\nu(z) + \nu K_\nu(z) = -zK_{\nu-1}(z)$$

4) $$\left(\frac{d}{zdz}\right)^m \left\{ z^\nu K_\nu(z) \right\} = (-1)^m z^{\nu-m} K_{\nu-m}(z)$$

5) $$\left(\frac{d}{zdz}\right)^m \left\{ \frac{K_\nu(z)}{z^\nu} \right\} = (-1)^m \frac{K_{\nu+m}(z)}{z^{\nu+m}}$$

6) $$K'_o(z) = -K_1(z)$$

7) $$K_{-\nu}(z) = K_\nu(z)$$

8) $$\frac{d}{dz}\left\{ z^\nu K_\nu(z) \right\} = z^\nu K_{\nu-1}(z)$$

9) $$\frac{d}{dx_1^m} [x_1^\nu K_\nu(k\, x_1)] = (-1)^m k^m x_1^\nu K_{\nu-m}(k\, x_1) \ .$$

Appendix IV: List of Fourier Transforms

All the following transforms are listed in Magnus, W., Oberhettinger, F., and Tricomi, F. G., Tables of Integral Transforms, California Institute of Technology, Bateman Manuscript Project, A. Erdelyi, editor (McGraw-Hill Book Co. 1954).

1) $$\int_0^\infty x^{\frac{1}{2}}(a^2 + x^2)^{-\frac{1}{2}} \, (xy)^{\frac{1}{2}} \, J_o(xy)dx = y^{-\frac{1}{2}}e^{-ay}$$ Re $a > 0$, $y > 0$

Vol.II, p.7, No.4

2) $$\int_0^\infty \frac{e^{-ax}}{x} \sin(xy)dx = \tan^{-1}(a^{-1}y)$$ Re $a > 0$, $y > 0$

Vol.I, p.8, No.11

3) $$\int_0^\infty \frac{\cos(xy)}{(x^2 + a^2)} \, dx = \frac{\pi}{2} \, a^{-1} \, e^{-ay}$$ Re $a > 0$, $y > 0$

Vol.I, p.8, No.18

4) $$\int_0^\infty J_o(xy)e^{-ax}dx = (y^2 + a^2)^{-\frac{1}{2}}$$ Re $a > 0$, $y > 0$

Vol.II, p.9, No.18

5) $$\int_0^\infty \sin(xy) \, \frac{x}{(a^2 + x^2)^{3/2}} \, dx = y \, K_o(ay)$$ Re $a > 0$, $y > 0$

Vol.I, p.66, No.27

6) $$\int_0^\infty xJ_o(xy)K_o(\beta x)dx = (\beta^4 + y^4 + 2\beta^2 y^2)^{-\frac{1}{2}}$$

Re $\beta > 0$

$y > 0$

7) $$\int_0^\infty x^{\nu+\frac{1}{2}}(x^2 + a^2)^{\mu-1}J_\nu(xy)(xy)^{\frac{1}{2}}dx = \frac{a^{\nu-\mu}y^{\mu+\frac{1}{2}}}{2^\mu\Gamma(\mu+1)} K_{\nu-\mu}(ay)$$

Re $a > 0$ $\quad y > 0$

$-1 <$ Re $\nu < 2$Re $\mu + 3/2$

Vol.II, p.24, No.20

8) $$\int_0^\infty x^{\pm\mu} K_\mu(ax)\cos(xy)dx = \frac{\sqrt{\pi}}{2}(2a)^{\pm\mu}\Gamma(\pm\mu + \tfrac{1}{2})(y^2 + a^2)^{\mp\mu-\frac{1}{2}}$$

Re $\mu > -\frac{1}{2}$ if upper signs are used

Re $\mu < \frac{1}{2}$ if lower signs are used

Re $a > 0$ $\quad y > 0$

Vol.I, p.49, No.41

9) $$\int_0^\infty \frac{\cos(xy)dx}{(a^2 + x^2)^{n+\frac{1}{2}}} dx = (\tfrac{1}{2}\,\tfrac{y}{a})^n[\Gamma(n+\tfrac{1}{2})]^{-1} K_n(ay)$$

Re $a > 0$, Re $n > -\frac{1}{2}$, $y > 0$

Vol.I, p.11, No.7

10) $$\int_0^\infty x^{-\lambda} K_\mu(ax)\sin(xy)dx$$

$$= y\Gamma(\tfrac{1}{2}\mu - \tfrac{1}{2}\lambda + 1)\Gamma(1 - \tfrac{1}{2}\lambda - \tfrac{1}{2}\mu)\ F\left(\frac{2+\mu-\lambda}{2}, \frac{2-\lambda-\mu}{2}; \frac{3}{2}; -\frac{y^2}{a}\right)$$

Re$(\lambda\pm\mu) < 2$, Re $a > 0$, $y > 0$

Vol.I, p.106, No.50

11) $$\int_0^\infty \frac{x^{2m}}{(x^2+a^2)^{\nu+\frac{1}{2}}} \cos(xy) = \frac{(-1)^m a^{-\nu}\sqrt{\pi}}{2^\nu \Gamma(\nu+\frac{1}{2})} \frac{d^{2m}}{dy^{2m}} [y^\nu K_\nu(ay)]$$

Re $a > 0$, $0 \leq m \leq$ Re $\nu + \frac{1}{2}$, $u > 0$

Vol.I, P.14, No.28

12) $$\int_0^\infty x^{-\lambda-\frac{1}{2}}(xy)^{\frac{1}{2}} J_\nu(xy) K_\mu(ax)dx$$

$$= \frac{\Gamma[\frac{1}{2}(\nu-\lambda+\mu+1)]\Gamma[\frac{1}{2}(\nu-\lambda-\mu+1)]}{2^{\lambda+1}a^{\nu-\lambda+1}\Gamma(\nu+1)y^{-\nu-\frac{1}{2}}} F\left(\frac{\nu-\lambda+\mu+1}{2}, \frac{\nu-\lambda-\mu+1}{2}; \nu+1; -\frac{y^2}{a^2}\right)$$

Re $a > 0$, $y > 0$, Re$(\nu-\lambda+1) > |\text{Re}(\mu)|$

Vol.II, p.63, No.4

13) $$\int_0^\infty \frac{x^{2m+1}}{(a^2+x^2)^{n+\frac{1}{2}}} \sin(xy)dx = \frac{(-1)^{m+1}\sqrt{\pi}}{2^n a^n \Gamma(n+\frac{1}{2})} \frac{d^{2m+1}}{dy^{2m+1}} [y^n K_n(ay)]$$

$-2 \leq 2m \leq 2n$, Re $a > 0$, $y > 0$

Vol.I, p.67, No.37

Appendix V: Some Properties of the Hypergeometric Functions $F(a,b;c;z)$

The hypergeometric functions are solutions of the differential equation

$$(1) \qquad z(1-z)\frac{d^2u}{dz^2} + [c - (a+b+1)z]\frac{du}{dz} - abu = 0 \ .$$

One of the solutions of (1) may be represented by a series which is regular at $z = 0$. This "hypergeometric series" is

$$(2) \qquad u = F(a,b;c;z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots \ .$$

When $c = -n$ $(n = 0,1,2,3,\ldots)$, the denominator of the $(n+2)$ term and the following all vanish and the series is undefined. However, the following representation is valid:

$$(3) \qquad \lim_{c=-n} \frac{F(a,b;c;z)}{\Gamma(c)} = \frac{a(a+1)\ldots(a+n)b(b+1)\ldots(b+n)F(a+n+1,\ b+n+1;\ n+2;\ z)}{(n+1)!}\, z^{n+1} \ .$$

The hypergeometric series converges on the unit circle $|z| = 1$ in the following cases:

a) $1 > \mathrm{Re}(a+b-c) \geq 0$ — convergence on the entire unit circle except for $z = 1$

b) $\quad Re(a+b-c) < 0 \quad$ absolute convergence on the entire unit circle including $z = 1$.

If $Re(a+b-c) \geq 1$ the series diverges on the entire unit circle. In all these cases we must assume that a, b, c are different from zero and, furthermore, that they are not negative integers.

A transformation formula for the hypergeometric function which is frequently used in this report is

$$(4) \qquad F(a,b;c;z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-z)^{-a} F(a,\ 1-c+a;\ 1-b+a;\ \frac{1}{z})$$

$$+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b} F(b,\ 1-c+b;\ 1-a+b;\ \frac{1}{z}) \ .$$

We also use

$$(5) \qquad \lim_{c \to -n} F(a,b;c;z) =$$

$$= \frac{a(a+1)\ldots(a+n)b(b+1)\ldots(b+n)z^{n+1}}{(n+1)!} F(a+n+1,b+n+1;n+2;z)$$

$$(6) \qquad \left\{ \begin{aligned} F(a,b;c;z) &= (1-z)^{-a} F(a,\ c-b;\ c;\ \frac{z}{z-1}) \\ &= (1-z)^{-b} F(b,\ c-a;\ c;\ \frac{z}{z-1}) \end{aligned} \right.$$

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